Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2017/2018

School of Mathematics and Statistics

BDIC1014J & BDIC1044J Linear Algebra

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MODULE COORDINATOR: Xin LIU
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Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry is written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge： ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Circle the preferred choice on the **Examination Book** provided.
This section is worth a total of **30** marks, with each question worth **3** marks.

1. For a linear system $\begin{cases} x - y + z = -2, \\ y - 2z = 2, \\ -x \quad z = 0, \end{cases}$ determine the number of its solution(s).

(a) unique solution; (b) two solutions; (c) inconsistent; (d) infinitely many solutions.

2. Let I be an $n \times n$ unit matrix, and A an $n \times n$ matrix satisfying $A^2 + A + I = O$. Then A^{-1} is

(a) $-(A+I)$; (b) $A+I$; (c) $A-I$; (d) $-(A-I)$.

3. If three $n \times n$ matrices A, B and C satisfy $AB = BA$ and $AC = CA$, then we have $ABC =$ __________.

(a) ACB; (b) CBA; (c) BCA; (d) CAB.

4. Given that

$$\begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 8 \\ 9 \end{pmatrix} - 3 \begin{pmatrix} -1 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 9 \\ -4 \\ 8 \\ 1 \end{pmatrix}$$

try to solve the unknowns (x, y, z) in the equation

$$\begin{pmatrix} 5 & -3 & 9 \\ -2 & 3 & -4 \\ 8 & -3 & 8 \\ 9 & 9 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$$

(a) $(1, 1, -1)$; (b) $(1, 3, -1)$; (c) $(-1, 3, 1)$; (d) $(-1, 1, 1)$.

5. Consider a matrix $\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & -3 & 9 & -27 \\ 5 & 6 & 7 & 8 \end{pmatrix}$. Let A_{4j} be the cofactor of $a_{4j},\ j = 1, 2, 3, 4$. Then if

$$A_{14} + aA_{24} + a^2A_{34} + a^3A_{44} = 0 \qquad \text{and} \qquad a > 0,$$

try to determine the value of $a =$__________.

(a) 1; (b) 3; (c) 5; (d) 7.

6. If $\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = 1$, try to compute $\det\begin{pmatrix} 4a_{11} & 2a_{11} - 3a_{12} & a_{13} \\ 4a_{21} & 2a_{21} - 3a_{22} & a_{23} \\ 4a_{31} & 2a_{31} - 3a_{32} & a_{33} \end{pmatrix} =$ __________.

(a) -12; (b) 12; (c) -24; (d) 24.

7. A matrix $A = \begin{pmatrix} k-1 & 2 \\ 2 & k-1 \end{pmatrix}$ is nonsingular, if and only if

(a) $k \neq -1$; (b) $k \neq 3$; (c) $k \neq -1$ and $k \neq 3$; (d) $k \neq -1$ or $k \neq 3$.

8. Given three vectors $\mathbf{v}_1 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 2 & 2 & 4 \end{pmatrix}^T$, $\mathbf{v}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, determine if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent or not, and choose a basis for the space they span.

(a) linearly independent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(b) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_3\}$;

(c) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$;

(d) linearly independent, and a basis is $\{\mathbf{v}_2, \mathbf{v}_3\}$.

9. *Cryptography*: Scherlock Holmes, a great detective, wished to send a message to the police to report a murderer's name.

- First, he used the following alphabet-number table

A	B	C	D	E	F	G	H	I	J	K	L	M	N
6	7	8	9	10	16	17	18	19	20	1	2	3	4
O	P	Q	R	S	T	U	V	W	X	Y	Z	space	
5	11	12	13	14	15	27	26	25	24	23	22	21	

  to encode a message into a string of 4 numbers, to form a column matrix M.

- Second, he used two *scramblers* (i.e., two invertible matrices) A_1 and A_2

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

  to encrypt M to be

$$A_2 A_1 M = M' = \begin{pmatrix} 10 \\ 1 \\ 8 \\ 6 \end{pmatrix}. \tag{1}$$

Then the numbers $\{10, 1, 8, 6\}$ were safely transmitted to the police.

Now, try to use A_1, A_2 and M' to recover the original message, to find out the name of the murderer.

(a) Jock; (b) Jane; (c) John; (d) Jack.

10. Let A be a 3×3 matrix. If A has the function

$$A \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} - 3a_{31} & a_{12} - 3a_{32} & a_{13} - 3a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

try to determine $A =$__________.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided. Only **brief** answers are needed. This section is worth a total of **20** marks, with each question worth **4** marks.

11. Suppose that A and B are two 3×3 matrices, with

$$\det A = 2, \qquad \det B = \frac{1}{2}.$$

Compute $\det\left(-2AB^T\right) =$ _______________.

12. Let A be an orthogonal matrix,

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

By definition, we immediately have the inverse $A^{-1} =$ __________.

13. Given that

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -2 & 3 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 6 & 1 \\ 9 & -1 & 4 & -4 \\ 8 & 2 & 6 & -1 \\ -1 & 7 & 2 & 1 \end{pmatrix},$$

try to compute

$$\begin{pmatrix} -2 & 1 & 1 & 3 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \underline{\qquad\qquad}, \qquad \begin{pmatrix} -2 & 1 & 1 & 3 \\ 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 1 & 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underline{\qquad\qquad}.$$

14. Consider a 5×5 matrix A, with five eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$ and $\lambda_5 = 5$. Try to evaluate the trace: $Tr\ A =$ _______________.

15. Consider a 5×5 matrix A, with five eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$ and $\lambda_5 = 5$.

If A is invertible, try to evaluate the determinant: $\det A^{-1} =$ _______________.

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **50** marks. The marks of each question are as shown.

16. (10 marks) Use three methods to solve the following linear system:

(a) row operations; *(3 marks)*

(b) inverse of the coefficient matrix, i.e., $\vec{x} = M^{-1}\vec{b}$; *(3 marks)*

(c) the Cramer's rule. *(4 marks)*

$$\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

17. (10 marks) Simply the following matrix, i.e., compute every entry of the matrix:

$$\begin{pmatrix}
+\begin{vmatrix} 0 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} \\
-\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
+\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} \\
-\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{vmatrix}
\end{pmatrix}.$$

18. (8 marks) (*Elementary matrices*) A transformation, say, $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$, can be realized by multiplying an elementary matrix, like:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}.$$

Find appropriate elementary matrices to realize the following transformations:

$$\text{(a)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 7 & 8 \\ 5 & 6 \\ 3 & 4 \\ 1 & 2 \end{pmatrix}; \quad \text{(b)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}; \quad \text{(c)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 6 & 5 \\ 8 & 7 \end{pmatrix}.$$

19. (8 marks)

(a) Use the geometric meaning of the following matrix to compute its power *(5 marks)*

$$\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}^{15}.$$

(b) The *Fibonacci sequence* are the numbers in the following integer sequence: *(3 marks)*

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots.$$

Denoting the i^{th} Fibonacci number as F_i, this sequence can be generated by the following recursive relation:

$$F_{n+2} = F_{n+1} + F_n, \qquad \text{with initial numbers } F_1 = F_2 = 1.$$

Try to express this recursive relation in terms of matrices, i.e., to find a matrix M such that

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = M \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

20. (14 marks) Let A be the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Use matrix diagonalization to prove

$$A^n = 3^{n-1} A, \qquad \text{where } n \in \mathbb{Z},\quad n > 0.$$

Glossary

Alphabet	字母表
Basis	基
Coefficient	系数
Cramer’s rule	克莱姆法则
Cryptography	密码学
Determinant	行列式
Diagonalization	对角化
Eigenvalue	本征值
Eigenvector	本征矢量
Elementary matrix	初等矩阵
Encode	编码
Encrypt	加密
Fibonacci sequence	斐波那契数列
Inverse	逆（矩阵）
Invertible	可逆
Linearly dependent	线性相关，线性依赖
Linearly independent	线性独立
Non-singular	非奇异
Orthogonal	正交
Recursive relation	递推关系
Row operations	行操作
Scrambler	加密矩阵
Unique	唯一