Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2016/2017

School of Mathematics and Statistics

BDIC1014J & BDIC1022J Linear Algebra

HEAD OF SCHOOL: Gary McGuire
MODULE COORDINATOR: Xin LIU

Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry is written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge： ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.
Circle the preferred choice on the **Examination Book** provided.
This section is worth a total of **45** marks, with each question worth **3** marks.

1. For a linear system $\begin{cases} x - y = -2, \\ y - z = 2, \\ -x + z = 0, \end{cases}$ determine the number of its solution(s).

(a) unique solution; (b) two solutions; (c) inconsistent; (d) infinitely many solutions.

2. Which of the following is true for all invertible matrices A and B of the same size:

(a) $(A - I)\left(A^{-1} + I\right) = A - A^{-1}$; (b) $(AB)^{-1} = A^{-1}B^{-1}$;

(c) $(A + B)(A - B) = A^2 - B^2$; (d) $(A - B)^2 = A^2 - 2AB + B^2$.

3. Which of the following is true for all 3×3 matrices A and B:

(a) $\det(AB) = \det A + \det B$; (b) $\det A^T = -\det A$;

(c) $\det(A + B) = \det A \det B$; (d) $\det(-A) = -\det A$.

4. Given that

$$2\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \\ -2 \\ 4 \end{pmatrix}$$

try to solve the unknowns (x, y, z) in the equation

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -1 \\ 3 \\ 1 \end{pmatrix}$$

(a) $(2, 1, -2)$; (b) $(1, 2, -2)$; (c) $(-2, 2, 1)$; (d) $(1, -2, 1)$.

5. Justify the following two steps of row operations acting on a matrix. (Notation below: $R_{\#}$ = Row #)

$$\begin{pmatrix} 0 & 1 & 2 & -1 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 7 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix} \overset{①}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 7 & 4 \\ 0 & 1 & 2 & -1 \end{pmatrix} \overset{②}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 1 & -2 \\ 0 & -4 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

(a) Step ①: $R_1 \to R_1 + R_4$, $R_4 \to R_2 - 3R_4$; Step ②: $R_3 \to R_3 - 4R_1$.

(b) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_3 \to R_3 - R_1 - R_2$.

(c) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_3 \to R_3 - 4R_1$.

(d) Step ①: $R_1 \to R_1 + R_4$, $R_4 \to R_2 - 3R_4$; Step ②: $R_3 \to R_3 - R_1 - R_2$.

6. Justify the following two steps of row or column operations acting on a determinant.
(Notations below: $R_{\#}$ = Row #, $C_{\#}$ = Column #)

$$\begin{vmatrix} -1 & 3 & -2 \\ -1 & 1 & -2 \\ 0 & 1 & 3 \end{vmatrix} \overset{①}{\Longrightarrow} \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix} \overset{②}{\Longrightarrow} \begin{vmatrix} 0 & 2 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

(a) Step ①: $C_1 \to -C_1$, $C_3 \to -C_3$; Step ②: $R_1 \to R_1 - R_2$.

(b) Step ①: $R_1 \to -R_1$, $R_2 \to -R_2$; Step ②: $R_1 \to R_1 - R_2$.

(c) Step ①: $C_1 \to -C_1$, $C_3 \to -C_3$; Step ②: $C_3 \to C_3 - 2C_1$.

(d) Step ①: $R_1 \to -R_1$, $R_2 \to -R_2$; Step ②: $C_3 \to C_3 - 2C_1$.

7. Find an elementary matrix to realize the following row reduction:

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

8. Which of the following is a *LU decomposition*?

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$;

(b) $\begin{pmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{pmatrix}$;

(c) $\begin{pmatrix} 1 & & \\ 4 & 1 & \\ 10 & 4 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{pmatrix}$;

(d) None of the above.

9. Given that a 3×3 matrix M has three distinct eigenvalues $2, 4, -5$, compute $\det M$.

(a) -1; (b) 1; (c) -40; (d) 40.

10. Evaluate the determinant: $\det \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}$.

(a) 4; (b) -32; (c) -16; (d) 16.

11. Determine the rank of the matrix $\begin{pmatrix} 4 & -4 & 12 & 8 \\ 1 & -1 & 3 & 2 \\ -1 & 1 & -3 & -2 \\ -3 & 3 & -9 & -6 \end{pmatrix}$.

(a) 1; (b) 2; (c) 3; (d) 4.

12. Consider two equations:

$$A: \qquad 3x^2 + 2xy + 3y^2 = 8,$$
$$B: \qquad 7x^2 + 2\sqrt{3}xy + 5y^2 = 16.$$

Given that

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\frac{1}{2}\begin{pmatrix} 7 & \sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

try to decide which of the following statements is correct.

(a) A describes an ellipse, and B a hyperbola.

(b) B describes an ellipse, and A a hyperbola.

(c) A and B both describe ellipses, and they have the same shape, i.e., they share the same long and short semi-axes.

(d) A and B both describe ellipses, but they have different shapes, i.e., they have different long and short semi-axes.

13. Given three vectors $\mathbf{v}_1 = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 3 & 1 & -2 \end{pmatrix}^T$, $\mathbf{v}_3 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$, determine if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent or not, and choose a basis for the space they span.

(a) linearly independent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(b) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(c) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$;

(d) linearly independent, and a basis is $\{\mathbf{v}_2, \mathbf{v}_3\}$.

14. *Cryptography*: Scherlock Holmes, a great detective, wished to send a message to the police to report a murderer's name.

- First, he used the following alphabet-number table

A	B	C	D	E	F	G	H	I	J	K	L	M	N
6	7	8	9	10	16	17	18	19	20	1	2	3	4
O	P	Q	R	S	T	U	V	W	X	Y	Z	space	
5	11	12	13	14	15	27	26	25	24	23	22	21	

  to encode a message into a string of 4 numbers, to form a column matrix M.

- Second, using two *scramblers* (i.e., two invertible matrices) A_1 and A_2

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

  he encrypted M to be

$$A_2 A_1 M = M' = \begin{pmatrix} 20 \\ 4 \\ 10 \\ 5 \end{pmatrix}. \tag{1}$$

Then the numbers $\{20, 4, 10, 5\}$ were safely transmitted to the police.

Now, try to use A_1, A_2 and M' to recover the original message, to find out the name of the murderer.

(a) Joan; (b) Joel; (c) John; (d) Joon.

15. In the above Question 14, can the *scramblers* A_1 and A_2 be replaced by the following matrices A_1' and A_2', respectively?

$$A_1' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A_2' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

(a) Yes for both A_1' and A_2'; (b) Yes for A_1', but No for A_2';

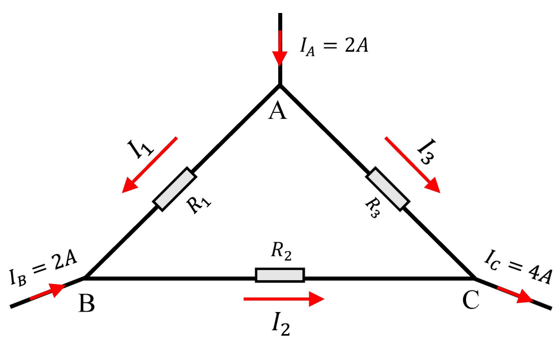
(c) No for A_1', but Yes for A_2'; (d) No for both A_1' and A_2'.

SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.

This section is worth a total of **55** marks. The marks of each question are as shown.

16. (10 marks) *Electric circuit and current*: Consider an electric circuit with three node points A, B and C, as shown below. The electric currents connecting to A, B and C are $I_A = 2A$, $I_B = 2A$ and $I_C = 4A$, measured in the unit Ampere (A). Let R_1, R_2 and R_3 be three electric resistances, measured in the unit ohm (Ω). Let I_1, I_2 and I_3 be the electric currents in between AB, BC and AC, respectively.



(a) Compute I_1, I_2 and I_3, by making use of I_A, I_B and I_C.
Can I_1, I_2 and I_3 be uniquely determined? *(4 marks)*

(b) Compute I_1, I_2 and I_3, when R_1, R_2 and R_3 are evaluated as follows: *(6 marks)*

$$R_1 = 1\Omega, \qquad R_2 = 1\Omega, \qquad R_3 = 2\Omega.$$

[**Hint*: The Ohm's law is $U = IR$, with U — electric voltage, I — current, R — resistance.]

17. (8 marks) Let $M = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

(a) Find the inverse of M by using the method of row operations. *(2 marks)*

(b) Find the inverse of M by using the method of adjoint matrix. *(3 marks)*

(c) Use the Cramer's rule to solve the linear system $\begin{cases} 2x + y = 4, \\ -x = 2. \end{cases}$ *(3 marks)*

18. **(14 marks)** Diagonalize $A = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, and evaluate A^5.

19. **(13 marks)** *Proofs & computations*:

(a) Let λ be an eigenvalue of a matrix A, with eigenvector $\mathbf{v}$. Prove λ^m is an eigenvalue of A^m with eigenvector $\mathbf{v}$, where $m = 1, 2, 3, \cdots$. *(4 marks)*

(b) Consider a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Try to **prove**

(i) Its characteristic equation for eigenvalues is *(3 marks)*

$$\lambda^2 - (\operatorname{tr} A)\,\lambda + \det A = 0 \qquad \text{(where "tr" denotes the operation Trace).}$$

(ii) The eigenvalues of A are real if and only if *(2 marks)*

$$\det A \leq \left(\frac{\operatorname{tr} A}{2}\right)^2.$$

(c) Evaluate a determinant *(4 marks)*

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

20. (10 marks) (*Matrix realization of differential operator*)

Sample: In order to realize a cyclic permutation of matrix components like $\begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ x^2 \\ x^3 \\ 1 \end{pmatrix}$,

we can take benefit of the following matrix left-multiplication:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} x \\ x^2 \\ x^3 \\ 1 \end{pmatrix}.$$

Using the idea of the above sample, try to:

(a) find a matrix A_1 to realize the functioning of the differential operator $\frac{d}{dx}$ in the sense of the following operation:

$$\frac{d}{dx} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}.$$

[**Hint*: Make use of the derivative formula $\frac{d}{dx} x^n = nx^{n-1}$.] *(5 marks)*

(b) find three matrices $A_2,\ A_3,\ A_4$ to realize the operators $\frac{d^2}{dx^2},\ \frac{d^3}{dx^3},\ \frac{d^4}{dx^4}$, respectively. Namely,

$$\frac{d^2}{dx^2} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = A_2 \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}. \qquad \frac{d^3}{dx^3} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = A_3 \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}. \qquad \frac{d^4}{dx^4} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix} = A_4 \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \end{pmatrix}.$$

What is the relationship between A_1 and the other $A_2,\ A_3,\ A_4$? *(5 marks)*

Glossary

Unique	唯一
Inconsistent	无解，不相容
Unknowns	未知数
Elementary matrix	初等矩阵
Distinct	互不相同的
Eigenvalue	本征值
Eigenvector	本征矢量
Rank	（矩阵的）秩
Ellipse	椭圆
Hyperbola	双曲线
Long (short) semi-axis	长（短）半轴
Linear independent	线性独立
Basis	基
Cryptography	密码学
Alphabet	字母表
Encode	编码
Scrambler	加密矩阵
Encrypt	加密
Current	电流
Resistance	电阻
Ampere	安培
Ohm	欧姆
Inverse	（矩阵的）逆
Diagonalization	对角化
Characteristic equation	示性方程
Differential operator	微分算子
Cyclic permutation	轮换；循环置换
Left-multiplication	左乘（一个矩阵）
Functioning	功能