Beijing-Dublin International College (BDIC)

FIRST SEMESTER, Academic Year 2015–2016

Campus: Beijing University of Technology (BJUT)

Linear Algebra — Final Exam

Honesty Pledge:

I have read and clearly understand the Examination Rules of Beijing University of Technology and University College Dublin and am aware of the Punishment for Violating the Rules of Beijing University of Technology and University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I would accept the punishment thereof.

Pledger: ______________________ Class NO: ______________________

BJUT Student ID: ________________ UCD Student ID: ________________

NOTE: Answer **ALL** questions.

Time allowed is **90** minutes.

The exam paper has **2** sections on **8** pages, with a full score of 100 marks.

You are required to use only the provided **Examination Book** for answers.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **45** marks, with each question worth **3** marks.

1. For a system of linear equations

$$\begin{cases} x & -2y & +2z & = \lambda, \\ x & -2y & -2z & = 2, \\ 2x & -4y & & = 0, \end{cases} \qquad (\lambda \in \mathbb{R})$$

determine the value of λ, such that the system is consistent (i.e., has at least one solution).

(a) 2; (b) -2; (c) no satisfactory value for λ; (d) any real number.

2. Which of the following is true for all invertible matrices A and B of the same size:

(a) $(A+I)\left(A^{-1}-I\right) = A^{-1} - A$;

(b) $(A+B)^{-1} = A^{-1} + B^{-1}$;

(c) $(A+B)(A-B) = A^2 - B^2$;

(d) $(A+B)^2 = A^2 + 2AB + B^2$.

3. Which of the following is true for all 2×2 matrices A and B:

(a) $\det(-A) = -\det A$;

(b) $\det A^T = -\det A$;

(c) $\det(A+B) = \det A + \det B$;

(d) $\det(AB) = \det A \det B$.

4. Determine the number of solutions for the following linear system:

$$\begin{cases} 2x & -y & +3z & = -1 \\ -x & +3y & -4z & = 2 \\ & 5y & -5z & = 9 \end{cases} .$$

(a) 1; (b) 2; (c) Infinitely many; (d) The system is inconsistent.

5. Justify the following two steps of row operations on a matrix. (Notation below: $R_\# = \text{Row } \#$)

$$\begin{pmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{pmatrix} \overset{①}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 2 & 3 & 6 & 2 \end{pmatrix} \overset{②}{\Longrightarrow} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -2 & -5 & 1 \\ 4 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \end{pmatrix}$$

(a) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \to R_2 - R_1 - R_4$, $R_4 \to R_4 - 2R_1$.

(b) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \to R_2 - 3R_1$, $R_4 \to R_4 - 2R_1$.

(c) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \to R_2 - 3R_1$, $R_4 \to 2R_4 - R_3$.

(d) Step ①: $R_1 \to \frac{1}{2}R_1$, $R_4 \to 2R_4$; Step ②: $R_2 \to R_2 - R_1 - R_4$, $R_4 \to R_4 - 2R_1$.

6. Justify the following two steps of row or column operations on a determinant.
(Notations below: $R_\# = \text{Row } \#$, $C_\# = \text{Column } \#$)

$$\begin{vmatrix} 3 & 6 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & 3 \end{vmatrix} \overset{①}{\Longrightarrow} \begin{vmatrix} 0 & 3 & 8 \\ 1 & 1 & -2 \\ 0 & 1 & 3 \end{vmatrix} \overset{②}{\Longrightarrow} \begin{vmatrix} 0 & 3 & -1 \\ 1 & 1 & -5 \\ 0 & 1 & 0 \end{vmatrix}$$

(a) Step ①: $C_1 \to 3C_1 - 2C_3$; Step ②: $C_3 \to C_3 - 3C_2$.

(b) Step ①: $C_1 \to 3C_1 - 2C_3$; Step ②: $C_3 \to C_3 - 3C_2$.

(c) Step ①: $R_1 \to R_1 - 3R_2$; Step ②: $C_3 \to C_3 - 3C_2$.

(d) Step ①: $R_1 \to R_1 - 3R_2$; Step ②: $R_3 \to 2R_3 + 3R_2$.

7. Find the elementary matrix to realize the following row reduction:

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 3 \\ 3 & 7 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 3 \\ 0 & 1 & 7 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$.

8. Which of the following is a *LU decomposition*?

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$;

(b) $\begin{pmatrix} 1 & & \\ 4 & 1 & \\ 10 & 4 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{pmatrix}$;

(c) $\begin{pmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -1 & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ & 1 & -1 \\ & & 1 \end{pmatrix}$;

(d) None of the above.

9. Given that a 3×3 matrix M has three distinct eigenvalues $3, 1, -1$, find its trace TrM.

(a) -3; (b) 3; (c) 0; (d) undeterminable.

10. Evaluate the determinant: $\det \begin{pmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 5 & -1 \\ 2 & 3 & 0 & 0 \\ 0 & -3 & 0 & 2 \end{pmatrix}$.

(a) 24; (b) 30; (c) 120; (d) -120.

11. With respect to the diagonalization: $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, we can make a transformation to a quadratic form

$$5x^2 - 4xy + 5y^2 = 3x'^{\,2} + 7y'^{\,2},$$

where the transformation between the coordinates (x, y) and (x', y') is given by

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x + y \\ -x + y \end{pmatrix}$; (b) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x' + y' \\ -x' + y' \end{pmatrix}$;

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x - y \\ x + y \end{pmatrix}$; (d) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x' - y' \\ x' + y' \end{pmatrix}$.

12. Given three vectors $\mathbf{v}_1 = \begin{pmatrix} 1 & 3 & -2 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 3 & 9 & -6 \end{pmatrix}^T$, $\mathbf{v}_3 = \begin{pmatrix} 2 & 5 & -7 \end{pmatrix}^T$, determine if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, and choose a basis for the space they span.

(a) linearly independent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(b) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$;

(c) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_3\}$;

(d) linearly independent, and a basis is $\{\mathbf{v}_2, \mathbf{v}_3\}$.

13. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors,

$$\mathbf{a} = \mathbf{i} + 4\mathbf{j}, \quad \mathbf{b} = -2\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}, \quad \mathbf{c} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Let $\mathbf{v} = \mathbf{i} + 4\mathbf{j} + \mathbf{k}$ be another vector. Try to find an appropriate triplet（三元组）of real numbers, (α, β, γ), to express $\mathbf{v}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$:

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}, \qquad \alpha, \beta, \gamma \neq 0.$$

(a) $(\frac{8}{5}, \frac{2}{5}, -\frac{1}{5})$; (b) $(\frac{8}{5}, -\frac{2}{5}, \frac{1}{5})$; (c) $(2, 1, -8)$;

(d) $\mathbf{v}$ is unable to be expressed by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

14. Determine the rank of the matrix $\begin{pmatrix} 5 & 2 & -1 & 3 \\ 1 & -4 & 3 & 2 \\ 6 & -2 & 2 & 5 \\ 4 & 6 & -4 & 1 \end{pmatrix}$.

(a) 1; (b) 2; (c) 3; (d) 4.

15. *Cryptography*（密码技术）: Suppose we have the following alphabet-number table

A	B	C	D	E	F	G	H	I	J	K	L	M	N
6	7	8	9	10	16	17	18	19	20	1	2	3	4
O	P	Q	R	S	T	U	V	W	X	Y	Z	space	
5	11	12	13	14	15	27	26	25	24	23	22	21	

Suppose we have a message which can be encoded into a string S, and then be written in rows to form a matrix M. Next, using the following matrix A [— which is invertible, called the *scrambler*（扰码矩阵/加密矩阵）],

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we can encrypt（加密）M to be

$$M' = \begin{pmatrix} 24 & 48 & 24 & 40 \\ 14 & 14 & 21 & 23 \\ 5 & 27 & 21 & 21 \end{pmatrix}. \qquad (1)$$

Then the rows of M' can be merged into a string

$$S' = \{24, 48, 24, 40, 14, 14, 21, 23, 5, 27, 21, 21\}$$

and be safely transmitted（传输）.

Try to use A and S' to recover the original message S.

(a) I HATE YOU; (b) I LOVE YOU; (c) I MISS YOU;

(d) YOU ARE IN DANGER.

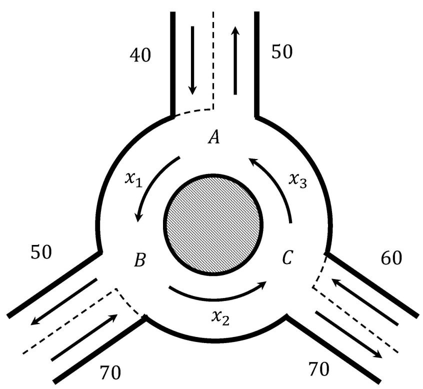
SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.

This section is worth a total of **55** marks, the marks of each question being as shown.

16. (7 marks) *Traffic control*: A roundabout（环岛）has three arms, A, B and C, as shown below. In one hour, the in-coming and out-going number of vehicles（车辆）at Arm A are respectively 40 & 50; those of B are 70 & 50; those of C are 60 & 70.

Use these data to determine the values of the traffic flows x_1, x_2 and x_3.



17. (8 marks) Find the inverse of the matrix

$$M = \begin{pmatrix} 2 & 6 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 3 \end{pmatrix}$$

by making use of the following two methods, respectively:

(a) elementary row operations; (*3 marks*)

(b) adjoint matrix. (*5 marks*)

Check if your results of (a) and (b) agree with each other.

18. (12 marks) Solve the system of linear equations

$$\begin{cases} 2x & +y & = 4 \\ -x & & = 2 \end{cases},$$

$$\text{i.e.,} \qquad A\vec{x} = \vec{b} \qquad \text{with} \qquad A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \ \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \ \text{and } \vec{b} = \begin{pmatrix} 4 \\ 2 \end{pmatrix},$$

by making use of the following three methods, respectively:

(a) elementary row operations on the augmented matrix $\left(A | \vec{b} \right)$; *(3 marks)*

(b) inverse matrix $A^{-1}\vec{b}$; *(3 marks)*

(c) Cramer's rule. *(6 marks)*

19. (13 marks) Diagonalize a matrix $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, and evaluate A^{2015}.

20. (15 marks) *Markov chain*（马尔可夫链）: Consider a *regular stochastic matrix*（随机矩阵）

$$M = \begin{pmatrix} p & q \\ 1-p & 1-q \end{pmatrix}, \qquad \text{where} \quad 0 < q < p < 1, \quad p, q \in \mathbb{R}.$$

M has two eigenvalues: one is 1, the corresponding eigenvector being a column vector $\mathbf{v}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $x_1 + x_2 = 1$; the other is $p - q$, the corresponding eigenvector being $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

A theorem about this regular stochastic matrix states

$$\lim_{n \to \infty} M^n = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_1 \end{pmatrix}. \tag{2}$$

$\mathbf{v}_1$ is called the *steady state vector*（稳态矢量）of M.

(a) Find $\mathbf{v}_1$, and verify the above limit behavior, eq.(2). *(12 marks)*

(b) Specially, when $p = \frac{1}{2}$ and $q = \frac{2}{5}$, i.e., $M = \begin{pmatrix} \frac{1}{2} & \frac{2}{5} \\ \frac{1}{2} & \frac{3}{5} \end{pmatrix}$, find $\mathbf{v}_1$, M^n and $\lim\limits_{n \to \infty} M^n$. *(3 marks)*