LINEAR ALGEBRA

(BDIC1014J / BDIC1044J)

Lecturers: *Xin LIU, Nicholas A. HOUSTON, Changjing ZHUGE*

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Beijing-Dublin International College (BDIC), Beijing University of Technology (BJUT)

Tutorial 7

LINEAR ALGEBRA PART — Eigenvalue problems (II)

1. Find all eigenvalues and eigenspaces for the following matrices:

   (a) $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; (b) $B = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$; (c) $C = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$.

2. Verify that if A is invertible and λ is an eigenvalue of A, then $\lambda \neq 0$ and λ^{-1} is an eigenvalue of A^{-1}. What can be said about eigenvalues of A^k where k is any integer?

3. Use the multiplicative property of the determinant to verify that if A and B are square matrices of the same size, and B is invertible, then A and $B^{-1}AB$ have the same eigenvalues.

4. Find the eigenvalues and corresponding eigenvectors for $M = \begin{pmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{pmatrix}$.

5. Suppose that $0 \leq \theta \leq \pi$. Verify that $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has real eigenvalues if and only if $\theta = 0$ or π.

6. Verify that a square matrix A has the same eigenvalues as its transpose A^T.

7. Let A be a square matrix with eigenvalue λ. Prove the following implications:

(a) $A^2 = 0 \implies \lambda = 0$;

(b) $A^2 = A \implies \lambda = 0$ or $\lambda = 1$;

(c) $A^2 = I \implies \lambda = 1$ or $\lambda = -1$.

8. Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Verify that the characteristic polynomial of A is

$$\lambda^2 - (a + d)\,\lambda + ad - bc.$$

Now also verify that

$$A^2 - (a + d)\,A + (ad - bc)\,I = 0.$$

This result says that, in matrix arithmetic, A is a root of its own characteristic polynomial, a special instance of the celebrated *Cayley-Hamilton* Theorem.

9. Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors for A;

(b) Write down an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

What is P^{-1}? What is D^n where n is any positive integer?

(c) Evaluate

$$A^n = PD^nP^{-1}.$$

for any positive integer n. Use your answer to find A^3 and A^4.

10. The matrix $B = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ has eigenvalues 2 and 4 with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

(a) Write down an invertible matrix P and a diagonal matrix D such that

$$B = PDP^{-1}.$$

(b) Find a formula for B^n, and use it to find B^3 and B^4.

11. The matrix $C = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvalues 0 , 1 and 3 with corresponding eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ respectively.

(a) Write down an invertible matrix P and diagonal matrix D such that

$$C = PDP^{-1}.$$

(b) Find a formula for C^n, and use it to find C^4.

12. Let $M = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(a) Find eigenvalues and corresponding eigenvectors for M;

(b) Write down an invertible matrix P and a diagonal matrix D such that

$$M = PDP^{-1};$$

(c) Evaluate

$$M^n = PD^nP^{-1}$$

for any positive integer n. Use your answer to find M^4.

13. Suppose $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are eigenvectors for a matrix M corresponding to different eigenvalues λ_1 and λ_2. Explain why $\boldsymbol{v}_1$ cannot be a scalar multiple of $\boldsymbol{v}_2$.

14. Three vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ are said to be *linearly independent* if

$$\alpha\boldsymbol{v}_1 + \beta\boldsymbol{v}_2 + \gamma\boldsymbol{v}_3 = \mathbf{0} \implies \alpha = \beta = \gamma = 0,$$

where α, β, γ are scalars. Explain why three eigenvectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ corresponding to three different eigenvalues λ_1, λ_2, λ_3 of a matrix M must be linearly independent.

15. Diagonalise $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ and find M^n for any positive integer n.

16. Diagonalise $M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$ and find M^n for any positive integer n.

17. Prove that $M = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is not diagonalisable.

18. The sequence of *Fibonacci numbers* is obtained by writing down 1 twice, and obtaining each successive number by adding the previous two numbers together:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots.$$

If we let x_n denote the nth Fibonacci number then

$$x_1 = x_2 = 1, \quad x_n = x_{n-1} + x_{n+2} \quad \text{for} \quad n \geq 3,$$

so that

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Diagonalise $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ to find a general formula for the nth Fibonacci number.