LINEAR ALGEBRA

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Tutorial 6

LINEAR ALGEBRA PART — Determinants & Eigenvalue problems (I)

1. Find the following determinants:

   (i) $\begin{vmatrix} 5 & 2 \\ 3 & -2 \end{vmatrix}$; (ii) $\begin{vmatrix} 6 & 2 \\ 3 & 1 \end{vmatrix}$; (iii) $\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$; (v) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{vmatrix}$;

   (iv) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{vmatrix}$; (v) $\begin{vmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ 3 & 3 & -6 \end{vmatrix}$; (vi) $\begin{vmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ -4 & 3 & 3 \end{vmatrix}$.

2. Find the determinant $\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix}$ by expanding along the first row.

3. Now find the determinant of the previous exercise by expanding

   (i) along the second row; (ii) along the third row; (iii) down the third column.

4. Write down quickly the determinants of the following matrices:

   (i) $\begin{vmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & -5 & -1 \end{vmatrix}$; (ii) $\begin{vmatrix} 3 & 3 & 8 \\ 0 & -6 & -7 \\ 0 & 0 & 2 \end{vmatrix}$; (iii) $\begin{vmatrix} -4 & -5 & 11 \\ 0 & 0 & 0 \\ 2 & -1 & 2 \end{vmatrix}$;

(iii) $\begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$; (iv) $\begin{vmatrix} 0 & 0 & 5 \\ 6 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix}$; (v) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ -6 & -3 & -7 & -1 \end{vmatrix}$.

5. Justify briefly the following calculation:

$$\begin{vmatrix} 2 & -3 & -2 \\ -1 & 3 & 4 \\ -7 & -2 & 8 \end{vmatrix} = \begin{vmatrix} 2 & -3 & -2 \\ 3 & -3 & 0 \\ 1 & -14 & 0 \end{vmatrix} = \begin{vmatrix} 2 & -1 & -2 \\ 3 & 0 & 0 \\ 1 & -13 & 0 \end{vmatrix} = -2 \begin{vmatrix} 3 & 0 \\ 1 & -13 \end{vmatrix} = 78$$

6. Use elementary row and column operations, or otherwise, to find the following:

(i) $\begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 3 \\ 4 & 5 & 1 \end{vmatrix}$; (ii) $\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$; (iii) $\begin{vmatrix} 2 & 3 & 6 & 2 \\ 3 & 1 & 1 & -2 \\ 4 & 0 & 1 & 3 \\ 1 & 1 & 2 & -1 \end{vmatrix}$.

7. Decide whether the following statements are true for all 2×2 matrices A and B:

(i) $\det(AB) = (\det A)(\det B)$; (ii) $\det(A+B) = (\det A) + (\det B)$;

(iii) $\det(2A) = 2\det A$; (iv) $\det(-A) = \det A$.

8. Use the multiplicative property of the determinant to verify that if A is an invertible matrix then $\det A \neq 0$ and $\det A^{-1} = (\det A)^{-1}$.

9. Let $\boldsymbol{u} = \boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k},\ \boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k},\ \boldsymbol{w} = -\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

(i) Compute $\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$; (ii) Use the result of (i) to compute $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$.

(iii) Explain how the formula $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \begin{vmatrix} u_1 & u_1 & u_1 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ follows from $\boldsymbol{u} \times \boldsymbol{v}$. Then use this formula to verify your result of (ii).

10. Determine the values of λ for which $\det(A - \lambda I) = 0$ in each case:

(i) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$; (iii) $A = \begin{pmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{pmatrix}$.

11. Verify that if A is a 2×2 or a 3×3 matrix then $\det A = \det A^T$ where A^T is the transpose of A, obtained by interchanging rows and columns ("reflecting in the diagonal").

12. For each of the following, compute $\det A$, $\operatorname{adj} A$ and A^{-1}:

(i) $\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$; (ii) $\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$; (iii) $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ -2 & 2 & -1 \end{pmatrix}$; (iv) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

13. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}.$$

(i) Compute the determinant of A. Is A nonsingular?

(ii) Compute $\operatorname{adj} A$ and the product $A \operatorname{adj} A$.

14. Show that:

(i) if A is nonsingular, then $\operatorname{adj} A$ is nonsingular and

$$(\operatorname{adj} A)^{-1} = \det\left(A^{-1}\right) A = \operatorname{adj}\left(A^{-1}\right);$$

(ii) if A is singular, then $\operatorname{adj} A$ is also singular.

15. Show that if $\det A = 1$, then

$$\operatorname{adj}(\operatorname{adj} A) = A.$$

16. Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that

$$\det(\operatorname{adj} A) = (\det A)^{n-1}.$$

17. Use Cramer's rule to solve each of the following systems:

(i) $\begin{cases} x_1 + 2x_2 = 3 \\ 3x_1 - x_2 = 1 \end{cases}$; (ii) $\begin{cases} 2x_1 + 3x_2 = 2 \\ 3x_1 + 2x_2 = 5 \end{cases}$;

(iii) $\begin{cases} 2x_1 & +x_2 & -3x_3 & = 0 \\ 4x_1 & +5x_2 & +x_3 & = 8 \\ -2x_1 & -x_2 & +4x_3 & = 2 \end{cases}$; (iv) $\begin{cases} x_1 & +x_2 & & & = 0 \\ & +x_2 & +x_3 & -2x_4 & = 1 \\ x_1 & & +2x_3 & +x_4 & = 0 \\ x_1 & +x_2 & & +x_4 & = 0 \end{cases}$.

18. Let B_j denote the matrix obtained by replacing the jth column of the identity matrix with a vector $\boldsymbol{b} = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}^T$. Use Cramer's rule to show that

$$b_j = \det B_j \qquad \text{for } j = 1, \cdots, n.$$

19. Find $A\boldsymbol{v}$ and $A\boldsymbol{w}$ where $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. By inspection, write down the two eigenvalues of A.

20. Find $B\boldsymbol{v}_1$, $B\boldsymbol{v}_2$ and $B\boldsymbol{v}_3$ where

$$B = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ \boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

By inspection, write down the three eigenvalues of B.

21. Find the characteristic polynomial $\det(M - \lambda I)$ and its roots in each case:

(i) $M = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$; (ii) $M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$; (iii) $M = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$.

22. Using answers from the above exercise, write down the eigenvalues of M in each case, and then find the corresponding eigenspaces.[*1]

23. Write down the eigenvalues immediately for the following triangular matrices, and then find all of the corresponding eigenspaces.

(i) $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; (ii) $M = \begin{pmatrix} 2 & 0 \\ -1 & -1 \end{pmatrix}$; (iii) $M = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{pmatrix}$.

[*1] To find an eigenspace corresponding to an eigenvalue means to find those eigenvectors corresponding to this eigenvalue and consequently forming this eigenspace.