LINEAR ALGEBRA

(BDIC1014J / BDIC1044J)

Lecturers: *Xin LIU, Nicholas A. HOUSTON, Changjing ZHUGE*

First Semester, Academic Year 2021–2022

Beijing-Dublin International College (BDIC), Beijing University of Technology (BJUT)

Tutorial 5

LINEAR ALGEBRA PART — Inverse matrices & Elementary matrices

1. Apply row reduction to invert the following 2×2 matrices:

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 5 & 4 \\ 3 & 3 \end{pmatrix}.$$

2. Solve each of the following systems by writing down an equivalent matrix equation and then multiplying through by A^{-1} or B^{-1} from the above exercise:

(i) $\begin{cases} x - 2y = 6 \\ -x + 3y = 5 \end{cases}$; (ii) $\begin{cases} x - 2y = -2 \\ -x + 3y = 1 \end{cases}$; (iii) $\begin{cases} 5x + 4y = -2 \\ 3x + 3y = -3 \end{cases}$.

3. Find the inverse of each of the following matrices when it exists:

(i) $\begin{pmatrix} 5 & 2 \\ 3 & -2 \end{pmatrix}$; (ii) $\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}$; (iii) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; (v) $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$;

(iv) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ -1 & 1 & 0 \end{pmatrix}$; (v) $\begin{pmatrix} 2 & 4 & 6 \\ 7 & 11 & 6 \\ -6 & -6 & 12 \end{pmatrix}$; (vi) $\begin{pmatrix} -4 & 3 & 3 \\ 8 & 7 & 3 \\ 4 & 3 & 3 \end{pmatrix}$.

4. Find the inverse of $\begin{pmatrix} 5 & -3 \\ 7 & -4 \end{pmatrix}$ and use it to solve for x, y, z, w where

$$\begin{pmatrix} 5 & -3 \\ 7 & -4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 11 & 4 \\ 15 & 5 \end{pmatrix}.$$

5. Find the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 3 \end{pmatrix}$ and use it to solve for x, y and z where

$$\begin{cases} x + y + z = 2 \\ 2x + 2y + 3z = 0 \\ 3x + 4y + 3z = 1 \end{cases}$$

6. Explain briefly why a square matrix with a row or column of zeros is not invertible.

7. Which of the following are true for all invertible matrices A, B, C of the same size:

   (i) $(ABC)^{-1} = A^{-1}B^{-1}C^{-1}$; (ii) $(ABA)^{-1} = A^{-1}B^{-1}A^{-1}$; (iii) $(A^{-1})^{-1} = A$;

   (iv) $-(-A)^{-1} = A^{-1}$; (v) $C^{-1}(ABC^{-1})^{-1}AB = I$; (vi) $(A + B)^{-1} = A^{-1} + B^{-1}$;

   (vii) $A^{-1}(I + A)A = A + I$; (viii) $(A + I)(A^{-1} - I) = A^{-1} - A$;

   (ix) $A^2 - 2A + I = 0 \implies A^{-1} = 2I - A$;

   (x) $A^2 - 2A + I = 0 \implies A = I$.

   Find a proof or counterexample in each case.

8. Use row reduction to determine the value of λ for which the following matrix is *not* invertible:

$$\begin{pmatrix} 1 & -2 & 3 \\ -3 & 1 & 2 \\ -3 & -4 & \lambda \end{pmatrix}$$

9. Suppose that M is an invertible matrix such that the inverse of $5M$ is $\begin{pmatrix} 5 & 6 \\ 5 & 5 \end{pmatrix}$. Find M.

10. Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ and solve for x, y, z in terms of a, b, c where

$$\begin{cases} x + 2y + 3z = a \\ 2x + 3y + z = b \\ 3x + y + 2z = c \end{cases}$$

11. When is a diagonal matrix $\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}$ invertible, and what is its inverse?

12. Let n be a positive integer and J the $n \times n$ matrix each of whose entries is 1. Verify that $I - J$ is invertible if and only if $n \geq 2$, in which case

$$(I - J)^{-1} = I - \frac{1}{n-1} J.$$

13. Use row reduction to determine the values of λ for which the matrix $A - \lambda I$ is *not* invertible in each case:

(i) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$; (iii) $A = \begin{pmatrix} -3 & 0 & 2 \\ -4 & -1 & 4 \\ -4 & -4 & 7 \end{pmatrix}$.

14. For each of the following pairs of matrices, find an elementary matrix E such that $EA = B$:

(i) $A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$, $B = \begin{pmatrix} -4 & 2 \\ 5 & 3 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 2 & 1 & 3 \\ -2 & 4 & 5 \\ 3 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 4 \\ -2 & 4 & 5 \end{pmatrix}$;

(iii) $A = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ -2 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 2 \\ 0 & 3 & 5 \end{pmatrix}$.

15. Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{pmatrix}.$$

(i) Find an elementary matrix E such that $EA = B$.

(ii) Find an elementary matrix F such that $FB = C$.

(iii) Is C row equivalent to A? Explain.

16. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{pmatrix}.$$

(i) Find elementary matrices E_1, E_2, E_3 such that

$$E_3 E_2 E_1 A = U$$

where U is an upper triangular matrix.

(ii) Determine the inverses of E_1, E_2, E_3 and set $L = E_1^{-1} E_2^{-1} E_3^{-1}$. What type of matrix is L? Verify that $A = LU$.

17. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 4 \\ 2 & 2 & 3 \end{pmatrix}$.

(i) Verify that $A^{-1} = \begin{pmatrix} 1 & 2 & -3 \\ -1 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}$.

(ii) Use A^{-1} to solve $A\boldsymbol{x} = \boldsymbol{b}$ for the following choices of $\boldsymbol{b}$:

(a) $\boldsymbol{b} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$; (b) $\boldsymbol{b} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^T$; (c) $\boldsymbol{b} = \begin{pmatrix} -2 & 1 & 0 \end{pmatrix}^T$.

18. Let

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}, \; C = \begin{pmatrix} 4 & -2 \\ -6 & 3 \end{pmatrix}.$$

Solve each of the following matrix equations:

(i) $AX + B = C$; (ii) $XA + B = C$;

(iii) $AX + B = X$; (iv) $AX + C = X$.

19. Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(i) Explain why U must be nonsingular.

(ii) Explain why U^{-1} must be upper triangular.

20. Let U_1 and U_2 be two $n \times n$ upper triangular matrices with nonzero diagonal entries. Show that $U_1 U_2$ is also an $n \times n$ upper triangular matrix.