LINEAR ALGEBRA

(BDIC1014J / BDIC1044J)

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Tutorial 4

LINEAR ALGEBRA PART — Chapter 4: Matrices and systems of equations

1. Use Gaussian elimination to solve the following linear systems.

(a) $\begin{cases} x_1 + x_2 + 2x_3 = 8 \\ -x_1 - 2x_2 + 3x_3 = 1 \\ 3x_1 - 7x_2 + 4x_3 = 10 \end{cases}$;

(b) $\begin{cases} x - y + 2z - w = -1 \\ 2x + y - 2z - 2w = -2 \\ -x + 2y - 4z + w = 1 \\ 3x - 3w = -3 \end{cases}$

2. For each of the following augmented matrices, write down a system of linear equations which is represented by the matrix

(a) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 3 \\ 0 & -1 & 2 & 4 \end{array}\right]$;

(b) $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{array}\right]$;

(c) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{array}\right]$.

3. Which of the following matrices are in row-echelon form?

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{bmatrix}$;

(e) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$;

(f) $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

4. Find an appropriate elementary row operation to describe the transformation from one matrix to the next.

(a) $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix}$;

(b) $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 6 & 8 \end{bmatrix}$;

(c) $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 0 & 1 \\ 0 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 1 \\ 3 & 1 & 4 \end{bmatrix}$.

5. Apply appropriate elementary row operations to the following augmented matrices to obtain row-echelon matrices. Solve the corresponding linear systems (assume the unknowns are x, y, z).

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 5 \\ 1 & 2 & 2 & -1 \\ 1 & 1 & 2 & -4 \end{array}\right]$;

(b) $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 3 & 6 & -1 & 12 \\ 2 & 4 & -1 & 7 \end{array}\right]$;

(c) $\left[\begin{array}{ccc|c} 1 & -1 & -3 & 2 \\ 1 & 0 & 1 & 0 \\ 2 & -1 & -2 & 3 \end{array}\right]$.

6. Which of the following matrices are in reduced row-echelon form?

(a) $\begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$;

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$;

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

(e) $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$;

(f) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

7. Solve the following systems of equations by writing down the associated augmented matrix and row reducing, either to echelon form and using back substitution, or to reduced echelon form (via the so-called Gauss-Jordan reduction) and reading off the answers:

(i) $\begin{cases} x + y - z = 0 \\ 2x - y + z = 9 \\ x + z = 10 \end{cases}$

(ii) $\begin{cases} -3x + 2y + z = 4 \\ 4x + y + 3z = 9 \\ x - y - z = -4 \end{cases}$

8. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 & 0 \\ -1 & -2 & 6 \end{bmatrix}, C = \begin{bmatrix} -3 & 5 & -1 \end{bmatrix}, D = \begin{bmatrix} 3 & 4 \\ 2 & 0 \\ 0 & -7 \\ 1 & -3 \end{bmatrix}$.

   Write down the sizes of A, B, C and D.

9. For the matrix

$$M = \begin{bmatrix} 6 & 0 & 3 & -5 \\ 0 & 7 & 2 & 4 \\ 1 & 3 & -2 & 0 \end{bmatrix}$$

   locate the

   (i) $(2,2)$-entry; (ii) $(3,3)$-entry; (iii) $(1,4)$-entry; (iv) $(3,2)$-entry; (v) $(3,4)$-entry.

10. Consider the following 2×2 matrices:

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ 4 & -3 \end{bmatrix}, D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

    Find:

    (i) $A + B$; (ii) $A - B$; (iii) $B - C$; (iv) $D + C$; (v) $2A$; (vi) $-B$; (vii) $\frac{1}{5}D$;

    (viii) AB; (ix) BA; (x) CD; (xi) BC; (xii) $A\,(BC)$; (xiii) $(AB)\,C$;

    (xiv) $ABCD$; (xv) A^2; (xvi) B^2; (xvii) $A^2 - B^2$; (xviii) $(A+B)(A-B)$.

11. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & 4 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}.$$

    (i) Find AB, BA, CD, DC and $BA + DC$.

    (ii) Explain briefly why A^2, B^2, C^2, D^2 and $AB + CD$ do not exist.

12. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$.

(i) Find AB, BA, CD and DC.

(ii) Simplify A^2B^2 and $C\left(DCDCD\right)^2C$ without any further matrix calculations.

13. Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}, X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, Y = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, Z = \begin{bmatrix} 5 & -2 \end{bmatrix}, W = \begin{bmatrix} 1 & 1 \end{bmatrix}$.
Find $AX, AY, ZA, WA, ZX, ZAX, ZY, ZAY, WX, WAX, WY, WAY$.

14. Find a 2×2 matrix M such that $M^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but every entry of M is nonzero.

15. Which of the following do you know to be true or expect to be true for all square matrices A, B, C of the same size:

(i) $(AB)C = A(BC)$; (ii) $AB = BA$; (iii) $(AB)^2 = A^2B^2$;

(iv) $A(B + C) = AB + AC$; (v) $(-A)(-B) = AB$;

(vi) $A(B - C) = AB - AC$; (vii) $(A + B)^2 = A^2 + 2AB + B^2$;

(viii) $(A + B)(A - B) = A^2 - B^2$; (ix) $(A + I)^2 = A^2 + 2A + I$;

(x) $(A + I)(A - I) = A^2 - I$.

Find a counterexample to each statement that you believe not to be true in general.

16. Consider the matrix
$$M = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}.$$

(i) Verify that $M^2 = 2M - I$.

(ii) Deduce that $M^3 = 3M - 2I$ and guess a general formula for powers of M . (If you know the technique of proof by induction then you can try to prove that your guess is correct.)

(iii) Evaluate M^5, M^{10} and M^{100}.