

# LINEAR ALGEBRA

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## Tutorial 2

### VECTOR PART — Chapter 2: Dot product

- Given that  $P = (8, 4, -1)$ ,  $Q = (6, 3, -4)$  and  $R = (7, 5, -5)$ , find  $\overrightarrow{QP}$ ,  $|\overrightarrow{QP}|$ ,  $\overrightarrow{QR}$ ,  $|\overrightarrow{QR}|$ ,  $\overrightarrow{QP} \cdot \overrightarrow{QR}$ , and the cosine of  $\angle PQR$ .
- Given that  $\mathbf{u} = i - 2j$  and  $\mathbf{v} = -2i + j$ , find
  - $\mathbf{u} \cdot \mathbf{v}$ ;
  - $\hat{\mathbf{u}}$ ;
  - $\hat{\mathbf{v}}$
  - $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$ ;
  - $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$ ;
  - $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ ;
  - $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$ ;
  - $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}$ ;
  - $\mathbf{v} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2} \mathbf{u}$ ;
  - $\mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v}$ ;
  - the cosine of the angle between  $\mathbf{u}$  and  $\mathbf{v}$ ;
  - the scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ ;
  - the scalar component of  $\mathbf{v}$  in the direction of  $\mathbf{u}$ ;
  - the vector projection of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ ;
  - the vector projection of  $\mathbf{v}$  in the direction of  $\mathbf{u}$ ;
  - the vector component of  $\mathbf{u}$  orthogonal to  $\mathbf{v}$ ;
  - the vector component of  $\mathbf{v}$  orthogonal to  $\mathbf{u}$ .
- Use the dot product to verify that if  $\mathbf{v}$  and  $\mathbf{w}$  are any vectors and  $\mathbf{w}$  is nonzero, then  $\mathbf{w}$  and  $\mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^2} \mathbf{w}$  are mutually perpendicular.
- Given that  $\mathbf{u} = i + 2j + 2k$  and  $\mathbf{v} = -4i + 4j - k$ , find
  - the cosine of the angle between  $\mathbf{u}$  and  $\mathbf{v}$ ;
  - the scalar component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ ;

- (iii) the scalar component of  $\mathbf{v}$  in the direction of  $\mathbf{u}$ ;
  - (iv) the vector projection of  $\mathbf{u}$  in the direction of  $\mathbf{v}$ ;
  - (v) the vector projection of  $\mathbf{v}$  in the direction of  $\mathbf{u}$ ;
  - (vi) the vector component of  $\mathbf{u}$  orthogonal to  $\mathbf{v}$ ;
  - (vii) the vector component of  $\mathbf{v}$  orthogonal to  $\mathbf{u}$ .
5. Verify that if  $\mathbf{a}$  and  $\mathbf{b}$  are vectors of the same length then  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{a} - \mathbf{b}$  are mutually perpendicular.
  6. Use vectors to find the following angles in a cube:
    - (i) between a major diagonal (between opposite vertices) and an edge;
    - (ii) between a major diagonal and a face diagonal;
    - (iii) between diagonals on adjacent faces;
    - (iv) between major diagonals.
  7. Use vectors to show that any angle inscribed in a semicircle is a right angle.
  8. Resolve the vector  $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 6\mathbf{k}$  into a sum of two vectors, one of which is parallel and the other perpendicular to  $\mathbf{u} = 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ .
  9. Find the components of the force  $15\mathbf{i} + 20\mathbf{j} + 6\mathbf{k}$  newtons in the direction of and orthogonal to
    - (i)  $-\mathbf{i} + \mathbf{j}$ ;
    - (ii)  $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ .
  10. Prove that if  $\mathbf{a}$  and  $\mathbf{b}$  are mutually perpendicular vectors then

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2.$$

Interpret this result in terms of a well-known fact about triangles.

11. Verify that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its sides.
12. Prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus (that is, has all sides of equal length).

VECTOR PART — Chapter 2: Cross product

13. Find two unit vectors perpendicular to both  $\mathbf{v}$  and  $\mathbf{w}$  where

$$\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 7\mathbf{k} \text{ and } \mathbf{w} = 5\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

14. Given that  $P = (8, 4, -1)$ ,  $Q = (6, 3, -4)$  and  $R = (7, 5, -5)$ , find

$$\overrightarrow{QP} \times \overrightarrow{QR}$$

and the area of the triangle  $\triangle PQR$ .

15. Given that  $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$ ,  $\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$  and  $\mathbf{c} = -2\mathbf{i} + \mathbf{k}$  find

(i)  $\mathbf{a} \times \mathbf{b}$ ;      (ii)  $\mathbf{a} \times \mathbf{c}$ ;      (iii)  $\mathbf{b} \times \mathbf{c}$ ;      (iv)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ ;      (v)  $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ ;

(vi)  $\mathbf{a} \times (\mathbf{a} \times \mathbf{c})$ ;      (vii)  $\mathbf{a} \times (\mathbf{a} + \mathbf{c})$ ;      (viii)  $(\mathbf{a} \times \mathbf{a}) \times \mathbf{c}$ ;      (ix)  $\mathbf{a} \times (\mathbf{b} - 2\mathbf{c})$ ;

(x) the sine of the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ;

(xi) the area of the parallelogram inscribed by  $\mathbf{a}$  and  $\mathbf{c}$ ;

(xii) the area of the triangle inscribed by  $\mathbf{b}$  and  $\mathbf{c}$ .

16. Given that  $\mathbf{v}$  and  $\mathbf{w}$  are vectors such that  $\mathbf{v} \times \mathbf{w} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ , find

(i)  $\mathbf{w} \times \mathbf{v}$ ;      (ii)  $(\mathbf{v} + 3\mathbf{w}) \times (2\mathbf{w} - \mathbf{v})$ .

17. Calculate  $|\mathbf{a} \times \mathbf{b}|$  given that  $|\mathbf{a}| = 7$ ,  $|\mathbf{b}| = 4$  and  $\mathbf{a} \cdot \mathbf{b} = -21$ .

18. A tetrahedron has four faces. Let  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  be vectors perpendicular to the faces, pointing outwards, of length equal to the respective areas of the faces. Verify that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}.$$

19. Verify that if  $\mathbf{a}$  and  $\mathbf{b}$  are geometric vectors then the following “correction” to the Cauchy-Schwarz Inequality holds:

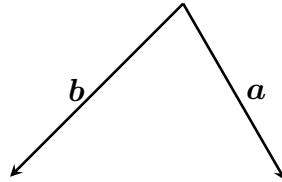
$$\sqrt{|\mathbf{a} \cdot \mathbf{b}|^2 + |\mathbf{a} \times \mathbf{b}|^2} = |\mathbf{a}| |\mathbf{b}|.$$

20. Find the areas of the triangles having vertices

(i)  $(0, 0, 0), (2, 2, -1), (3, -4, 2)$ ;

(ii)  $(3, -1, 2), (1, -1, -3), (4, -3, 1)$ .

21. Let  $\mathbf{a}$  and  $\mathbf{b}$  be the following vectors in the page:



True or false:

(i)  $\mathbf{a} \times \mathbf{b}$  points upwards, away from the page, towards the ceiling;

(ii)  $\mathbf{b} \times (\mathbf{a} - \mathbf{b})$  points downwards, away from the page, towards the floor;

(iii)  $\mathbf{a} \times (\mathbf{b} \times \mathbf{a})$  is perpendicular to  $\mathbf{a}$  but not to  $\mathbf{b}$ ;

(iv)  $\mathbf{b} \times (\mathbf{b} \times \mathbf{a})$  is the zero vector.

22. Use the cross product to find

(i) a unit vector perpendicular to both  $-\mathbf{i} + 2\mathbf{j}$  and  $\mathbf{j} + 3\mathbf{k}$ ;

(ii) a unit vector which points in a direction perpendicular to the triangle with vertices  $A(0, 0, -1)$ ,  $B(1, -2, -1)$  and  $C(1, -3, -4)$  such that looking backwards along the vector (from tip to tail) towards the triangle, the vertices  $A, B, C$  rotate anticlockwise (in that order).

23. Verify that, for any geometric vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$ ,

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

24. Verify that, for any geometric vectors  $\mathbf{a}, \mathbf{b}$  and  $\mathbf{c}$ ,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Use anti-commutativity to deduce that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$