LINEAR ALGEBRA

(BDIC1014J / BDIC1044J)

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First Semester, Academic Year 2021–2022

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Tutorial 1

VECTOR PART — Appendices A, B & C

1. Let $I = \{1, 2, \cdots, 10\}$, $A = \{1, 3, 5, 8, 9\}$, $B = \{2, 4, 6, 8, 9\}$ and $C = \{5, 6, 8, 10\}$. Find:

   (1) $A \cup B$; (2) $A \cap B$; (3) $A \backslash B$; (4) $\overline{C}$; (5) $\overline{A \cup C}$ and $\overline{A} \cap \overline{C}$;
   (6) $\overline{B \cap C}$ and $\overline{B} \cup \overline{C}$; (7) $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$;
   (8) $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$; (9) $(A \backslash C) \cup C$ and $A \cup C$; (10) $(B \backslash C) \cap C$.

2. Let two sets be $A = \left\{ (x, y) \middle| \frac{y-3}{x-2} = a + 1 \right\}$ and $B = \{ (x, y) | (a^2 - 1) x + (a - 1) y = 15 \}$, with $a \in \mathbb{R}$. If $A \cap B = \varnothing$, find the value of a.

3. Prove: (1) $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$; (2) $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

4. Evaluate: $\sum_{i=0}^{n} [a + (i - 1) d]$; $\sum_{i=1}^{n} [a + (i - 1) d]$ (with $a, d > 0, a, d \in \mathbb{R}$).

5. (a) Evaluate $5 + 9 + 13 + 17 + \cdots + 61 + 65$.

   (b) Express the following sum in big sigma notation: $15x + 13x^3 + 11x^5 + \cdots - 5x^{21}$.

   (c) Evaluate $\sum_{k=2}^{29} \left(\frac{1}{k} - \frac{1}{k+1}\right)$.

6. (a) Evaluate $\sum_{i=0}^{5} \frac{1}{3^i}$.

   (b) Write the sum of the first n terms of the sequence $5, 10, 20, 40, \cdots$ in sigma notation. Evaluate this sum for $n = 20$.

7. Show that $k \cdot k! = (k+1)! - k!$. Hence find $\sum_{k=1}^{50} k \cdot k!$.

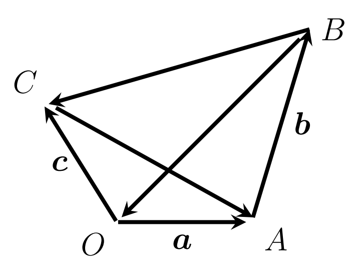
8. Let a be a real number. Suppose the following formula holds for all positive integers $n \in \mathbb{Z}^+$
$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n^2 + an}{4(n+1)(n+2)}.$$
Then determine the value of a.

9. Prove by mathematical induction that
$$n! > 3^n$$
for all integers $n \geq 7$.

10. Let $f(n)$ be a function of a positive integer n, $f(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. When $n \geq 2$, prove
$$n + f(1) + f(2) + \cdots + f(n-1) = nf(n).$$

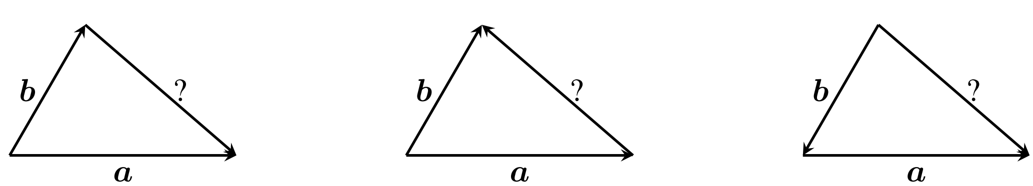
VECTOR PART — Chapter 1

11. Suppose we have the following vectors in two dimensions as shown. Let us denote $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{AB} = \boldsymbol{b}$ and $\overrightarrow{OC} = \boldsymbol{c}$. Find the expressions for $\overrightarrow{CA}$, $\overrightarrow{BO}$ and $\overrightarrow{BC}$ in terms of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$.



12. Let $\overline{ABCDEF}$ be a regular hexagon and put $\boldsymbol{a} = \overrightarrow{AB}$, $\boldsymbol{b} = \overrightarrow{BC}$. Find vector expressions in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$ for the displacements $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{EF}$ and $\overrightarrow{FA}$.

13. In each diagram below, find the unknown vector in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.



14. Let $\boldsymbol{v} = 2\boldsymbol{i} - 6\boldsymbol{j} + 9\boldsymbol{k}$ and $\boldsymbol{w} = 4\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$. Find

(i) $-\boldsymbol{v}$; (ii) $\boldsymbol{w} - \boldsymbol{v}$; (iii) $2\boldsymbol{v}$; (iv) $3\boldsymbol{w}$; (v) $2\boldsymbol{v} - 3\boldsymbol{w}$;
(vi) $|\boldsymbol{v}|$; (vii) $|\boldsymbol{w}|$; (viii) $\hat{\boldsymbol{v}}$; (ix) $\hat{\boldsymbol{w}}$; (x) $|\boldsymbol{v} + \boldsymbol{w}|$.

15. Given points $A(4, -1, 5)$ and $B(6, -1, -2)$ in space, find

   (a) the position vectors of A and B in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$;

   (b) the displacement vector $\overrightarrow{AB}$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$;

   (c) the unit vector pointing from A towards B;

   (d) the unit vector pointing from B towards A.

16. A balloon experiences two forces, a buoyancy force of 8 newtons vertically upwards and a wind force of 6 newtons acting horizontally to the right. Calculate the magnitude and direction of the resultant force.

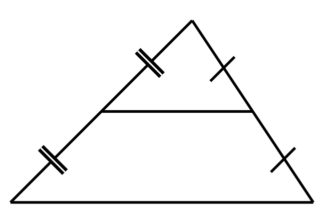
17. Explain, geometrically, the *triangle inequality*

$$|\boldsymbol{v} + \boldsymbol{w}| \leq |\boldsymbol{v}| + |\boldsymbol{w}|,$$

and determine when equality occurs.

18. Let $\boldsymbol{v} = \overrightarrow{PQ}$ where $P = (-3, 2, 0)$ and $Q = (4, -2, 3)$. Find the Cartesian form of $\boldsymbol{v}$, the length of $\boldsymbol{v}$ and the angles $\boldsymbol{v}$ makes (to the nearest degree) with each of the positive x-, y- and z-axes. (The cosine of each angle will be the relevant component divided by the length of the vector.)

19. Prove, using vectors, that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half the length of the third side.



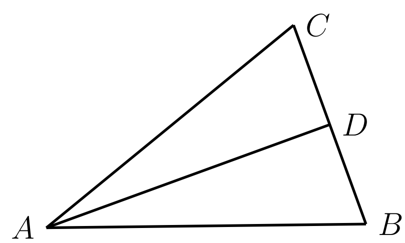
20. Express $2\boldsymbol{a} - 3\boldsymbol{b}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$, and simplify, when $\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{v},\ \boldsymbol{b} = 3\boldsymbol{u} - 2\boldsymbol{v}$.

21. Suppose that $\boldsymbol{v}$ and $\boldsymbol{w}$ are two non-zero vectors which are not parallel and the following vector equation holds for some scalars α and β:

$$\boldsymbol{v} + \alpha(\boldsymbol{w} - \boldsymbol{v}) = \beta\left(\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}\right).$$

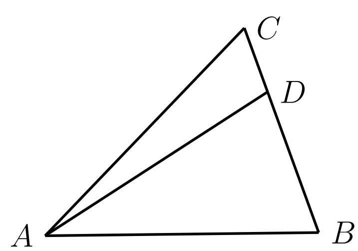
Find α and β.

22. Let D be the midpoint of the side BC of the triangle $\triangle ABC$.



Verify that $\overrightarrow{AD} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right)$.

23. Let D be the point which divides the side BC of the triangle $\triangle ABC$ in the ratio $\alpha : \beta$.



Verify that

$$\overrightarrow{AD} = \frac{\beta\overrightarrow{AB} + \alpha\overrightarrow{AC}}{\alpha + \beta}.$$

24. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ denote displacements of 1 km east and north respectively. An aeroplane travels 300 km southeast and then 150 km in the direction $30°$ west of north. Find

    (a) the above displacements of the aeroplane and their vector sum in terms of the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$;

    (b) the final distance (to the nearest km) and direction (to nearest degree, south of east) of the aeroplane from the starting position.