

Beijing-Dublin International College (BDIC)

FIRST SEMESTER, Academic Year 2015–2016

Campus: Beijing University of Technology (BJUT)

Linear Algebra — Final Exam (Supplemental)

Honesty Pledge:

I have read and clearly understand the Examination Rules of Beijing University of Technology and University College Dublin and am aware of the Punishment for Violating the Rules of Beijing University of Technology and University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I would accept the punishment thereof.

Pledger: _____

Class NO: _____

BJUT Student ID: _____

UCD Student ID: _____

NOTE: Answer **ALL** questions.

Time allowed is **90** minutes.

The exam paper has **2** sections on **8** pages, with a full score of 100 marks.

You are required to use only the provided **Examination Book** for answers.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **50** marks, with each question worth **5** marks.

1. Which of the following is true for all invertible matrices (可逆矩阵) A and B of the same size:

- (a) $AB = BA$; (b) $(AB)^{-1} = A^{-1}B^{-1}$; (c) $(ABA^{-1})^2 = A^2B^2A^{-2}$;
 (d) $(A + A^{-1})^2 = A^2 + 2I + A^{-2}$.

2. Which of the following is true for all 3×3 matrices A and B :

- (a) $\det(-A) = \det A$; (b) $\det A^T = -\det A$;
 (c) $\det(A + B) = \det A + \det B$; (d) $\det(AB) = \det A \det B$.

3. Determine the number of solutions for the following linear system:

$$\begin{cases} x & & +z & = 3 \\ -x & +2y & +3z & = 1 \\ x & +2y & +5z & = 7 \end{cases} .$$

- (a) 1; (b) 2; (c) Infinitely many; (d) The system is inconsistent (无解).

4. Justify the following two steps of elementary row operations (初等行变换) on a matrix.

(Notation below: $R_{\#} = \text{Row } \#$)

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 3 & 4 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \xrightarrow{\textcircled{1}} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & 0 & -1 \\ -1 & 3 & 4 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{\textcircled{2}} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 4 & 3 & 0 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

- (a) Step ①: $R_1 \leftrightarrow R_3$; Step ②: $R_2 \rightarrow R_2 + 2R_3$, $R_3 \rightarrow R_3 + R_1$.
 (b) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \rightarrow R_2 + 2R_3$, $R_3 \rightarrow R_3 + R_1$.
 (c) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 + R_1$.
 (d) Step ①: $R_1 \rightarrow R_1 - R_3$; Step ②: $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 + R_1$.

5. Find the elementary matrix (初等矩阵) to realize the following row operation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$.

6. Which of the following is a *LU decomposition* (LU分解) ?

$$(a) \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix};$$

$$(b) \begin{pmatrix} 1 & & \\ 4 & 1 & \\ 10 & 4 & 1 \end{pmatrix} = \begin{pmatrix} & 1 & \\ & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{pmatrix};$$

$$(c) \begin{pmatrix} 1 & 2 & 4 \\ -2 & -3 & -2 \\ 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ & 1 & 6 \\ & & 1 \end{pmatrix};$$

(d) None of the above.

7. A 3×3 matrix M has three distinct eigenvalues (本征値) $5, 0, -1$. Find its trace (迹) TrM .

(a) 4;

(b) 0;

(c) -5 ;

(d) 5.

8. Evaluate the determinant (行列式) : $\det \begin{pmatrix} 2 & 0 & 0 \\ -7 & 4 & 0 \\ 3 & 2 & 5 \end{pmatrix}$.

(a) 0;

(b) 11;

(c) -40 ;

(d) 40.

9. We can make a transformation to a quadratic form (二次型)

$$3x^2 + 2xy + 3y^2 = 2x'^2 + 4y'^2,$$

where the transformation between the coordinates (x, y) and (x', y') is given by _____.

(* *Hint* (提示) :

$$\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.)$$

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -x + y \\ x + y \end{pmatrix};$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x + y \\ x + y \end{pmatrix};$

(c) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -x' + y' \\ x' + y' \end{pmatrix};$

(d) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x' - y' \\ x' + y' \end{pmatrix}.$

10. Solve the following linear system

$$\begin{cases} 3x + 6y + 2z = 2, \\ x + y - 2z = 4, \\ y + 3z = -1. \end{cases}$$

$$(\text{Hint: } \det \begin{pmatrix} 3 & 6 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix} = -1;$$

$$\det \begin{pmatrix} 2 & 6 & 2 \\ 4 & 1 & -2 \\ -1 & 1 & 3 \end{pmatrix} = -40;$$

$$\det \begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & -2 \\ 0 & -1 & 3 \end{pmatrix} = 22;$$

$$\det \begin{pmatrix} 3 & 6 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} = -7.)$$

- (a) $x = \frac{1}{7}$, $y = \frac{40}{7}$, $z = -\frac{22}{7}$.
 (b) $x = \frac{1}{40}$, $y = \frac{1}{-22}$, $z = \frac{1}{7}$.
 (c) $x = 40$, $y = -22$, $z = 7$.
 (d) $x = -41$, $y = 21$, $z = -8$.

SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.

This section is worth a total of **50** marks, the marks of each question being as shown.

11. (10 marks) Let \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{v} be four column vectors:

$$\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}; \quad \mathbf{v} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}.$$

Try to find a set of real numbers α, β, γ ($\alpha, \beta, \gamma \neq 0$) to linearly express \mathbf{v} ,

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}.$$

12. (12 marks) Find the inverse (逆) of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

by making use of the following two methods, respectively:

- (a) elementary row operations (初等行变换); (5 marks)
- (b) adjoint matrix (伴随矩阵). (7 marks)

Check if your results of (a) and (b) agree with each other.

13. (13 marks) Solve the system of linear equations

$$\begin{cases} 4x + 3y = 1 \\ 5x - 2y = 7 \end{cases},$$

i.e., $A\vec{x} = \vec{b}$ with $A = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$,

by making use of the following two methods, respectively:

- (a) elementary row operations on the augmented matrix (增广矩阵) $(A|\vec{b})$; (6 marks)
- (b) Cramer's rule (克莱姆法则). (7 marks)

14. (15 marks) Diagonalize (对角化) the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and evaluate A^{2016} .