Beijing-Dublin International College (BDIC)

FIRST SEMESTER, Academic Year 2014–2015

Campus: Beijing University of Technology (BJUT)

Linear Algebra — Final Exam (Supplemental)

Honesty Pledge:

I have read and clearly understand the Examination Rules of Beijing University of Technology and University College Dublin and am aware of the Punishment for Violating the Rules of Beijing University of Technology and University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I would accept the punishment thereof.

Pledger: ____________________ Class NO: ____________________

BJUT Student ID: ______________ UCD Student ID: ______________

NOTE: Answer **ALL** questions.

Time allowed is **90** minutes.

The exam paper has **2** sections on **8** pages, with a full score of 100 marks.

You are required to use only the provided **Examination Book** for answers.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **45** marks, with each question worth **3** marks.

1. Consider three points $A\,(1,0,0)\,, B\,(0,1,0)$ and $C\,(0,0,1)$ in 3 dimensions. Which of the following is an equation describing the plane passing through these points?

(a) $x+y+z=1$; (b) $x+y+z=3$; (c) $x-1=y-1=z-1$;

(d) $(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=0$.

2. Consider the points A, B, C given in Question (1). Find the vector component of $\overrightarrow{AB}$ parallel to the vector $\overrightarrow{AC}$.

(a) $\frac{1}{2}\,(\mathbf{k}-\mathbf{i})$; (b) $\frac{1}{2}\,(\mathbf{i}+\mathbf{j}+\mathbf{k})$; (c) $\frac{1}{\sqrt{2}}$; (d) $\frac{1}{\sqrt{2}}\,(\mathbf{j}-\mathbf{i})$.

3. Which of the following is true for all invertible matrices A and B of the same size:

(a) $AB=BA$; (b) $(AB)^{-1}=A^{-1}B^{-1}$; (c) $\left(ABA^{-1}\right)^2=A^2B^2A^{-2}$;

(d) $\left(A+A^{-1}\right)^2=A^2+2I+A^{-2}$.

4. Which of the following is true for all 3×3 matrices A and B:

(a) $\det(-A)=\det A$; (b) $\det A^T=-\det A$;

(c) $\det(A+B)=\det A+\det B$; (d) $\det(AB)=\det A\det B$.

5. Determine the number of solutions for the following linear system:

$$\begin{cases} x & & +z & = 3 \\ -x & +2y & +3z & = 1 \\ x & +2y & +5z & = 7 \end{cases}.$$

(a) 1; (b) 2; (c) Infinitely many; (d) The system is inconsistent.

6. Justify the following two steps of row operations on a matrix. (Notation below: $R_{\#} = \text{Row } \#$)

$$\begin{pmatrix} 0 & -1 & 1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 3 & 4 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \overset{①}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 2 & 1 & 0 & -1 \\ -1 & 3 & 4 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix} \overset{②}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & 4 & 3 & 0 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

(a) Step ①: $R_1 \leftrightarrow R_3$; Step ②: $R_2 \to R_2 + 2R_3$, $R_3 \to R_3 + R_1$.

(b) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \to R_2 + 2R_3$, $R_3 \to R_3 + R_1$.

(c) Step ①: $R_1 \leftrightarrow R_4$; Step ②: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

(d) Step ①: $R_1 \to R_1 - R_3$; Step ②: $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$.

7. Justify the following two steps of row or column operations on a determinant.
(Notations below: $R_{\#} = \text{Row } \#$, $C_{\#} = \text{Column } \#$)

$$\begin{vmatrix} 3 & 6 & 2 \\ 1 & 2 & -2 \\ 0 & 1 & 3 \end{vmatrix} \overset{①}{\Longrightarrow} \begin{vmatrix} 3 & 0 & 2 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix} \overset{②}{\Longrightarrow} \begin{vmatrix} 4 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{vmatrix}$$

(a) Step ①: $C_2 \to C_2 - 2C_1$; Step ②: $C_3 \to 2C_3 - 3C_1$.

(b) Step ①: $C_2 \to C_2 - 2C_1$; Step ②: $R_1 \to R_1 + R_2$.

(c) Step ①: $R_1 \to R_1 - 3R_2$; Step ②: $R_1 \to R_1 + R_2$.

(d) Step ①: $R_1 \to R_1 - 3R_2$; Step ②: $C_3 \to 2C_3 - 3C_1$.

8. Find the elementary matrix to realize the following row reduction:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$; (c) $\begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$.

9. Which of the following is a *LU decomposition*?

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ & 1 \end{pmatrix}$;

(b) $\begin{pmatrix} 1 & & \\ 4 & 1 & \\ 10 & 4 & 1 \end{pmatrix} = \begin{pmatrix} & & 1 \\ & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{pmatrix}$;

(c) $\begin{pmatrix} 1 & 2 & 4 \\ -2 & -3 & -2 \\ 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$;

(d) None of the above.

10. Given that a 3×3 matrix M has three distinct eigenvalues $5, 0, -1$, find its trace TrM.

(a) 4; (b) 0; (c) -5; (d) 5.

11. Evaluate the determinant: $\det\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & -7 & 4 & 0 \\ 0 & 3 & 2 & 5 \\ 3 & 9 & 11 & 2 \end{pmatrix}$.

(a) -33; (b) -120; (c) 120; (d) 40.

12. With respect to the diagonalization: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, we can make a transformation to a quadratic form

$$3x^2 + 2xy + 3y^2 = 2x'^{\,2} + 4y'^{\,2},$$

where the transformation between the coordinates (x, y) and (x', y') is given by

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -x + y \\ x + y \end{pmatrix}$; (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x + y \\ x + y \end{pmatrix}$;

(c) $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -x' + y' \\ x' + y' \end{pmatrix}$; (d) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x' - y' \\ x' + y' \end{pmatrix}$.

13. Given three vectors $\mathbf{v}_1 = \begin{pmatrix} 1 & 3 & 3 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}^T$, $\mathbf{v}_3 = \begin{pmatrix} 4 & 2 & -2 \end{pmatrix}^T$, determine if $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, and choose a basis for the space they span.

(a) linearly independent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$;

(b) linearly dependent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_2\}$;

(c) linearly dependent, and a basis is $\{\mathbf{v}_2, \mathbf{v}_3\}$;

(d) linearly independent, and a basis is $\{\mathbf{v}_1, \mathbf{v}_3\}$.

14. Determine the rank of the matrix $\begin{pmatrix} 5 & 2 & -1 & 3 \\ 1 & -4 & 3 & 2 \\ 6 & -2 & 2 & 5 \\ 4 & 6 & -4 & 1 \end{pmatrix}$.

(a) 1; (b) 2; (c) 3; (d) 4.

15. Solve the following linear system

$$\begin{cases} 3x & +6y & +2z & = 2, \\ x & +y & -2z & = 4, \\ & y & +3z & = -1. \end{cases}$$

(*Hint:* $\det\begin{pmatrix} 3 & 6 & 2 \\ 1 & 1 & -2 \\ 0 & 1 & 3 \end{pmatrix} = -1$; $\det\begin{pmatrix} 2 & 6 & 2 \\ 4 & 1 & -2 \\ -1 & 1 & 3 \end{pmatrix} = -40$;

$\det\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & -2 \\ 0 & -1 & 3 \end{pmatrix} = 22$; $\det\begin{pmatrix} 3 & 6 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} = -7$.)

(a) $x = \frac{1}{7}, \quad y = \frac{40}{7}, \quad z = -\frac{22}{7}$.

(b) $x = \frac{1}{40}, \quad y = \frac{1}{-22}, \quad z = \frac{1}{7}$.

(c) $x = 40, \quad y = -22, \quad z = 7$.

(d) $x = -41, \quad y = 21, \quad z = -8$.

SECTION B — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.

This section is worth a total of **55** marks, the marks of each question being as shown.

16. (11 marks) (11 marks) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors,

$$\mathbf{a} = -\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

(a) Compute $\mathbf{b} \times \mathbf{c}$. *(2 marks)*

(b) Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$, and determine the volume of the parallelepiped （平行六面体）which has three adjacent edges represented $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively. *(3 marks)*

(c) Let $\mathbf{v} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ be another vector. Try to express $\mathbf{v}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, i.e., to find an expression for $\mathbf{v}$:

$$\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c},$$

where α, β, γ are not all zeroes, $\alpha, \beta, \gamma \in \mathbb{R}$. *(6 marks)*

17. (8 marks) Find the inverse of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

by making use of the following two methods, respectively:

(a) elementary row operations; *(3 marks)*

(b) adjoint matrix. *(5 marks)*

Check if your results of (a) and (b) agree with each other.

18. Solve the system of linear equations

$$\begin{cases} 4x & +3y & = 1 \\ 5x & -2y & = 7 \end{cases},$$

$$\text{i.e.,} \qquad A\vec{x} = \vec{b} \qquad \text{with} \qquad A = \begin{pmatrix} 4 & 3 \\ 5 & -2 \end{pmatrix},\ \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 7 \end{pmatrix},$$

by making use of the following two methods, respectively:

(a)

(b) elementary row operations on the augmented matrix $\left(A|\vec{b}\right)$; (*5 marks*)

(c) Cramer's rule. (*6 marks*)

19. (13 marks) Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and evaluate A^{2015}.

20. (12 marks) Consider the matrix $M = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} \\ \frac{3}{4} & \frac{7}{8} \end{pmatrix}$, whose entries are positive and the columns add to 1. M is an example of a *regular stochastic matrix* （正则随机矩阵）. M has two eigenvalues: one is $\frac{1}{8}$, with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$; the other is 1, with eigenvector denoted by $\mathbf{v}$. The components of $\mathbf{v}$ add up to 1.

A theorem about regular stochastic matrices states

$$\lim_{n\to\infty} M^n = \begin{bmatrix} \mathbf{v} & \mathbf{v} \end{bmatrix} \qquad \text{(where } \mathbf{v} \text{ is a column vector).} \tag{1}$$

Hence $\mathbf{v}$ is called the *steady state vector* （稳态矢量）of M.

Find $\mathbf{v}$, and verify the above limit behavior, as shown in eq.(1), for the given M.

Linear Algebra

EXAMINATION BOOK

Pledger: ____________________ Class NO: ____________________

BJUT Student ID: ______________ UCD Student ID: ______________

SECTION A

Circle the preferred answer.

If you make a mistake, mark a cross through your wrong choice and circle your next alternative.

1.	*a*	*b*	*c*	*d*
2.	*a*	*b*	*c*	*d*
3.	*a*	*b*	*c*	*d*
4.	*a*	*b*	*c*	*d*
5.	*a*	*b*	*c*	*d*
6.	*a*	*b*	*c*	*d*
7.	*a*	*b*	*c*	*d*
8.	*a*	*b*	*c*	*d*
9.	*a*	*b*	*c*	*d*
10.	*a*	*b*	*c*	*d*
11.	*a*	*b*	*c*	*d*
12.	*a*	*b*	*c*	*d*
13.	*a*	*b*	*c*	*d*
14.	*a*	*b*	*c*	*d*
15.	*a*	*b*	*c*	*d*