Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2020/2021

BDIC1014J & BDIC1044J Linear Algebra

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Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The marks that each question carry are written as shown.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge： ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. NO dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE CHOICE QUESTIONS

In each question, choose **at most one** option.

Circle the preferred choice on the **Examination Book** provided.

This section is worth a total of **20** marks, with each question worth **4** marks.

1. A matrix $M = \begin{pmatrix} k & k \\ k & 1 \end{pmatrix}$ is nonsingular, if and only if

(a) $k = 0$; (b) $k = 1$; (c) $k \neq 1$ and $k \neq 0$; (d) $k \neq 1$ or $k \neq 0$.

2. Let A and B be two invertible $n \times n$ matrices. Which of the following statements are always true?

(a) $(AB)^2 = A^2B^2$;

(b) $\operatorname{tr}\left(AB^{-1}\right) = \operatorname{tr}\left(B^{-1}A\right)$;

(c) $(A + B)^2 = A^2 + 2AB + B^2$;

(d) $\det(-AB) = -\det(AB)$.

3. For a linear system $\begin{cases} x - y - z = 1, \\ x - y + z = 1, \\ y + z = 2, \end{cases}$ determine the number of solution(s):

(a) inconsistent; (b) unique solution; (c) two solutions; (d) infinitely many solutions.

4. Every 2×2 matrix M can be associated to a unique vector via

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longrightarrow \quad \boldsymbol{m} = \begin{pmatrix} a & b & c & d \end{pmatrix}^T,$$

so that M can be written in terms of a basis of four linearly independent vectors,

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T \right\}.$$

For example, for $M = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$, *we have* $\boldsymbol{m} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + 5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

Question: how many linearly independent vectors are needed, if M is required to be *singular*?

(a) 1; (b) 2; (c) 3; (d) 4.

5. Find a correct expression for the unknowns x or y in

$$\begin{cases} 5x & +7y & = & -2, \\ 9x & -4y & = & 1. \end{cases}$$

(a) $x = \dfrac{\begin{vmatrix} 5 & 7 \\ 9 & -4 \end{vmatrix}}{\begin{vmatrix} -2 & 7 \\ 1 & -4 \end{vmatrix}}$; (b) $x = -\dfrac{\begin{vmatrix} 5 & -2 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 9 & -4 \end{vmatrix}}$; (c) $y = \dfrac{\begin{vmatrix} -2 & 7 \\ 1 & -4 \end{vmatrix}}{\begin{vmatrix} 5 & -2 \\ 9 & 1 \end{vmatrix}}$; (d) $y = \dfrac{\begin{vmatrix} 5 & -2 \\ 9 & 1 \end{vmatrix}}{\begin{vmatrix} 5 & 7 \\ 9 & -4 \end{vmatrix}}$.

SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided. Only **brief** answers are needed.

This section is worth a total of **30** marks, with each question worth **3** marks.

6. Consider an invertible matrix $M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Let A_{ij} be the cofactor of $a_{ij},\ i,j = 1,2$. Then:

(a) $a_{11}A_{11} + a_{12}A_{12} =$ ______________ (options: $\det M$, 0, $\frac{1}{\det M}$ or undetermined);

(b) $a_{11}A_{21} + a_{12}A_{22} =$ ______________ (options: $\det M$, 0, $\frac{1}{\det M}$ or undetermined).

7. For the matrix

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -1 & 1 & -2 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix},$$

compute

$$\operatorname{rank} A = \text{\_\_\_\_\_\_\_\_\_\_}.$$

8. Considering a matrix M with characteristic polynomial

$$\lambda^2 - 3\lambda + 2,$$

compute

$$\det M = \text{\_\_\_\_\_\_\_\_\_\_}, \qquad \operatorname{tr} M = \text{\_\_\_\_\_\_\_\_\_\_}.$$

9. Considering the vectors

$$\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j}, \qquad \boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{k},$$

compute

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \underline{\hspace{3cm}}.$$

10. If A is an orthogonal matrix, compute

$$\det A = \underline{\hspace{3cm}}.$$

11. Let A and B be two 2×2 matrices,

$$A = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}.$$

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two 2-vectors satisfying

$$\boldsymbol{y} = AB^{-2}\boldsymbol{x}.$$

Compute the angle of rotation from $\boldsymbol{x}$ to $\boldsymbol{y}$: $\underline{\hspace{3cm}}$.

12. Let A and B be two 3×3 matrices, where A is invertible and

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}.$$

For $\alpha \in \mathbb{R}$, compute

$$\operatorname{tr}\left[A(B - \alpha I)A^{-1}\right] = \underline{\hspace{3cm}}.$$

13. Compute the following matrix multiplication

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \underline{\qquad\qquad\qquad}.$$

14. If X is an invertible matrix and $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, the eigenvalues of XAX^{-1} are $\underline{\qquad\qquad\qquad}$.

15. Consider two matrices A and B, where $A = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$, and B is a symmetric matrix satisfying

$$\boldsymbol{x}^T A\, \boldsymbol{x} = \boldsymbol{x}^T B\, \boldsymbol{x}\,, \qquad\qquad \forall\, \boldsymbol{x} \in \mathbb{R}^2.$$

Compute the matrix $B = \underline{\qquad\qquad\qquad}$.

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **50** marks. The marks of each question are as shown.

16. (7 marks) Consider a 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Prove that A is *invertible*, i.e., *non-singular*. *(3 marks)*

(b) Find the inverse of A by using two methods: *(4 marks)*

the elementary row operation (ERO) and *the adjoint matrix* .

17. (9 marks)

(a) Let A and B be two 4×4 matrices, where

$$A = (\boldsymbol{\alpha}, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4), \qquad B = (\boldsymbol{\beta}, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4)$$

with $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{v}_2$,$\boldsymbol{v}_3$, $\boldsymbol{v}_4$ being four 4-dimensional column vectors. If

$$\det A = 4, \qquad \det B = 1,$$

then try to compute the determinant $\det(A + B)$. *(4 marks)*

(b) Consider $\det M = \begin{vmatrix} 13 & 0 & 14 & 0 \\ 2 & 2 & 2 & 3 \\ 0 & -17 & 0 & 0 \\ 5 & 3 & -2 & 2 \end{vmatrix}$. Let A_{41}, A_{42}, A_{43} and A_{44} be the cofactors corresponding to the bottom row entries, i.e., 5, 3, -2 and 2, respectively. Compute the sum

$$A_{41} + A_{42} + A_{43} + A_{44}.$$ *(5 marks)*

18. (11 marks) Let $A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

(a) Solve the linear system $A\boldsymbol{x} = \boldsymbol{b}$ by the method of elementary row operations.

Notice: *If there exist free variables, you should choose the variables with the largest subscripts as the free ones, i.e., in the order* $x_4,\ x_3,\ \cdots$. *(5 marks)*

(b) Are the four column vectors of A, i.e., $\boldsymbol{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{\alpha}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\boldsymbol{\alpha}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{\alpha}_4 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, linearly independent or dependent? Explain your answer. *(3 marks)*

(c) Determine whether the vector $\boldsymbol{b}$ can be expressed by a linear combination of $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$. If yes, give the linear combination; if no, explain why. *(3 marks)*

19. (10 marks) *Lower-upper (LU) decomposition.* Consider a matrix $A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 0 \\ 1 & 4 & 1 \end{pmatrix}$. Find two 3×3 matrices L and U such that

$$A = LU,$$

where

- L is a lower triangular matrix;
- U is an upper triangular matrix, with all the diagonal entries being 1, i.e.,

$$U = \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix}, \qquad * \text{ — a real number.}$$

20. (13 marks) Consider a 3×3 matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Find the eigenvalue(s) of A, and the eigenvector(s) corresponding to every eigenvalue.

Specify the *algebraic* and *geometric multiplicities* of every eigenvalue. (*6 marks*)

Note: Algebraic multiplicity — number of repetition of a root.

Geometric multiplicity — number of the eigenvectors of a root.

(b) Find a matrix X and its inverse X^{-1}, to diagonalize A as

$$X^{-1}AX = D, \qquad \text{with } \lambda_1 \geq \lambda_2 \geq \lambda_3.$$ (*5 marks*)

(c) Evaluate the power of the matrix, A^{2020}. (*2 marks*)

[Note]:

- *No need to compute the exact value of a number to the power of* 2020.
- *No need to compute the exact value of the final matrix multiplication. It is sufficient to write the result as the product of three matrices.*

Glossary

Adjoint matrix	伴随矩阵
Algebraic multiplicity	代数重数
Angle of rotation	旋转角
Characteristic polynomial	示性多项式
Cofactor	代数余子式
Diagonalization	对角化
Eigenvalue	本征值，特征值
Eigenvector	本征矢量，特征矢量
Elementary matrix	初等矩阵
Elementary row operation (ERO)	初等行变换
Geometric multiplicity	几何重数
Inconsistent	无解，不相容
Infinitely many	无穷多
Invertible	可逆的
Linear combination	线性组合
Linearly dependent	线性相关，线性依赖
Linearly independent	线性无关，线性独立
Lower-upper (LU) decomposition	（矩阵的）上下分解
Nonsingular	非奇异的，（矩阵）可逆的
Orthogonal	正交
Rank	秩
Repetition	重复
Specify	说明，解释
Subscript	脚标
Sufficient	充分的
Symmetric	对称的
Triangular	三角的
Unique	唯一的
Unknown	未知数