Beijing-Dublin International College

SEMESTER I FINAL EXAMINATION – 2021/2022

School of Mathematics and Statistics
BDIC1014J & BDIC1044J Linear Algebra

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Time Allowed: 90 minutes

Instructions for Candidates

Answer ALL questions. The exam paper has **3** sections, with a full score of **100** marks.

BJUT Student ID: ________________ **UCD Student ID:** ________________

I have read and clearly understand the Examination Rules of both Beijing University of Technology and University College Dublin. I am aware of the Punishment for Violating the Rules of Beijing University of Technology and/or University College Dublin. I hereby promise to abide by the relevant rules and regulations by not giving or receiving any help during the exam. If caught violating the rules, I accept the punishment thereof.

Honesty Pledge： ____ **(Signature)**

Instructions for Invigilators

Non-programmable calculators are permitted. No dictionaries are permitted.
No rough-work paper is to be provided for candidates.

SECTION A — MULTIPLE-CHOICE QUESTIONS

Each question has only **<u>ONE</u>** correct solution.
Write your answers on the **Examination Book** provided.
This section is worth **24** marks in total, with each question worth **4** marks.

1. Let

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}, \qquad C = \begin{pmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_4+b_4 & a_5+b_5 & a_6+b_6 \\ a_7+b_7 & a_8+b_8 & a_9+b_9 \end{pmatrix}.$$

Consider two formulae:

P1: $A + B = C, \qquad \forall a_i, b_i \in \mathbb{R};$

P2: $\det A + \det B = \det(A + B), \qquad \forall a_i, b_i \in \mathbb{R}.$

Determine which of the following statements is correct.

(a) P1 and P2 are both true. (b) P1 is true but P2 is false.

(c) P1 is false but P2 is true. (d) P1 and P2 are both false.

2. Which of the following is an elementary matrix?

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

3. Let A be a 3×3 matrix, and λ its unique eigenvalue, $\lambda \neq 0$.

Determine which of the following statements is FALSE.

(a) The matrix $2\lambda I - A$ is invertible.

(b) The rank of $\lambda I - A$ must be zero.

(c) $\det A = \lambda^3$.

(d) Let $\boldsymbol{x}$ be an eigenvector. Then $A\boldsymbol{x} = \lambda\boldsymbol{x}$ as a linear system must have infinitely many solutions.

4. Let A be a real symmetric matrix. Which of the following statements is NOT correct?

(a) A is positive definite, if and only if every eigenvalues of A is positive.

(b) A is positive definite, if and only if $\boldsymbol{x}^T A\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{x} \neq \boldsymbol{0}$.

(c) A is positive definite, if and only if $\boldsymbol{x}^T A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^m$.

(d) A is positive definite, if there exists an orthogonal matrix P such that $A = P^T DP$, with D a diagonal matrix with all the diagonal entries being positive real numbers.

5. Let λ_1 and λ_2 be two distinct eigenvalues of a square matrix A, and $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ the corresponding eigenvectors, respectively. Determine which of the following statements is CORRECT.

(a) $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must be linear independent.

(b) $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ must be linear dependent.

(c) Unable to determine the linear dependence/independence of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

(d) None of above.

6. Consider an arbitrary matrix $A = \left[\begin{pmatrix} \boldsymbol{v}_1 \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_2 \end{pmatrix} \cdots \begin{pmatrix} \boldsymbol{v}_n \end{pmatrix}\right]$, where $\boldsymbol{v}_j,\ j = 1, \cdots, n$, are the column vectors of A. Letting $\boldsymbol{b}$ be a column vector with constant entries, we can use A and $\boldsymbol{b}$ to construct a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{x}$ is the vector of unknowns.

Then, if the $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n$ are linearly independent, which of the following statements is CORRECT?

(a) The linear system has at most one solution.

(b) The linear system has at least one solution.

(c) The linear system has exactly one solution.

(d) None of the above is correct.

SECTION B — GAP-FILLING QUESTIONS

Write your answers on the **Examination Book** provided. Only **brief** answers are needed. This section is worth a total of **35** marks, with each question worth **5** marks.

7. Consider the following vectors in three dimensions:

$$\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \qquad \boldsymbol{b} = -2\boldsymbol{i} + 2\boldsymbol{k}, \qquad \boldsymbol{c} = \boldsymbol{i} + \boldsymbol{j} + \beta\boldsymbol{k}.$$

If they are linearly dependent, the value of the constant $\beta =$ ____________.

8. Let $f(x) = \begin{vmatrix} 0 & 1 & 0 \\ x & 9 & -3 \\ 1 & 5 & 1 \end{vmatrix}$. The derivative of $f(x)$ is given by $f'(x) =$ ____________.

9. Let A be a square matrix and λ its eigenvalue. If $A^2 = 0$, then $\lambda =$ ____________.

10. Let E_1, E_2, E_3 be three elementary matrices, and A be a 3×3 matrix. If $E_1E_2E_3A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$,

then $\operatorname{rank} A =$ ____________.

11. For $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, try to find

$A^{-1} =$ ____________, $\det A =$ ____________, $\operatorname{adj} A =$ ____________.

12. Consider a determinant $\begin{vmatrix} 3 & 2 & 1 \\ -3 & 5 & 9 \\ 4 & 2 & -6 \end{vmatrix}$. Let A_{21}, A_{22} and A_{23} denote the cofactors of the entries $a_{21} = -3$, $a_{22} = 5$ and $a_{23} = 9$, respectively. Try to compute $3A_{21} - 2A_{22} + A_{23} =$ ____________.

13. Compute the power $A = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}^{2021} =$ ____________.

SECTION C — EXTENDED ANSWER QUESTIONS

Write your answers on the **Examination Book** provided.
This section is worth a total of **41** marks. The marks of each question are as shown.

14. (6 marks) Let $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ be three linearly independent vectors. Determine the linear dependence/independence of $\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_2+\boldsymbol{\alpha}_3$ and $\boldsymbol{\alpha}_3+\boldsymbol{\alpha}_1$, and **prove your conclusion**.

15. (7 marks) Consider a linear system about three unknowns x_1, x_2 and x_3,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3, \end{cases} \qquad \text{where } a_{ij}\text{'s and } b_i\text{'s are constants.}$$

If $\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, and the matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}$ is non-singular, try to solve the unknowns x_1, x_2 and x_3.

Requirement: Express your answer in numbers.

16. (9 marks) For $A = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$, perform the LU-decomposition,

$$A = LU,$$

where L and U are, respectively, a lower and an upper triangular matrix, with all the diagonal entries of U being 1.

17. **(8 marks)** Let A_n denote the $n \times n$ determinant

$$A_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a_1 & 0 & \cdots & 0 \\ 1 & 0 & a_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & a_{n-1} \end{vmatrix}$$

where $a_i \neq 0$ and $a_i \neq 1$ for $i = 1, 2, \cdots, n$. Answer the following questions.

(a) Show that for every $n > 2$,

$$A_n = a_{n-1}A_{n-1} - a_1a_2 \cdots a_{n-2}.$$

(b) Calculate the value of A_2.

(c) Use the method of mathematical induction to prove

$$A_n = a_1a_2 \cdots a_{n-1}\left(1 - \sum_{i=1}^{n-1} \frac{1}{a_i}\right), \qquad \forall n \geq 2.$$

18. **(11 marks)** Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

(a) Calculate the eigenvalues and the corresponding eigenvectors.

Requirement: The norms of the eigenvectors should be 1.

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$ where the diagonal entries of D should be sorted descend from top left to bottom right.

Glossary

Arbitrary	任意的
Bottom right	右下角
Cofactor	代数余子式
Column (vector)	列（矢量）
Conclusion	结论，结果
Constant	常数
Derivative	导数，微商
Descend	降（序）
Determinant	行列式
Diagonal matrix	对角矩阵
Distinct	不同的
Eigenvalue	特征值，本征值
Eigenvector	特征向量，本征矢量
Elementary matrix	初等矩阵
Entry	矩阵元
Formulae	公式
Infinitely many	无穷多
Invertible	可逆的
Linear system	线性方程组
Linearly dependent	线性相关，线性依赖
Linearly independent	线性无关，线性独立
Lower triangular matrix	下三角矩阵
LU-decomposition	LU 分解
Mathematical induction	数学归纳法
Non-singular	非奇异
Norm	（矢量的）长度，模长
Orthogonal matrix	正交矩阵
Positive definite	正定的
Real symmetric matrix	实对称矩阵
Solution	解
Sort	排序
Square matrix	方矩阵
Statement	陈述，说法
Top left	左上角
Unique	唯一
Unknown	未知数
Upper triangular matrix	上三角矩阵